Bölüm 3

Blow-up for Some Semilinear Parabolic Problems With Generalized Sources

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Abstract

In this paper, we establish some conditions on parameter β to guarantee the existence of non-global solutions for the semilinear parabolic problems involving generalized sources with local and nonlocal type reaction terms and derive the upper bounds for the blow-up time and a criterion for blow-up. Moreover, we obtain the bounded of solutions under the suitable condition on parameter β.

1 Introduction

There are many articles that are concerned with nonlinear evolutionary processes and examine the behavior of the solution to initial or initial-boundary value problems that model the process. Because of the nonlinearity, some problems that emerge in ohmic heating, chemical reactions, gaseous ignition, and chemotaxis in biological systems exhibit explosive growth of the solution. Studies regard with these problems are generally related to the existence or nonexistence of global solutions and the blow-up of the solution in finite time. In some papers, various conditions and criteria for nonlinearity that imply blow-up occurs were presented and bounds on the blow-up time, structure of the blow-up, and the asymptotic behavior of the solution were determined (see [5, 10, 15] and references therein). According to Bandle and Brunner, the physical meaning of an explosion is generally thought of as a very large increase in temperature that leads to the combustion of a chemical reaction. In [23], there is a large bibliography examining the explosive behavior of solutions to problems in mechanics (see also [22]). In [12], the author considered various properties

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of the solutions of a large class of mixed initial-boundary value problems for the parabolic equation. This paper has been an important resource for researchers studying regard with blow-up in reaction-diffusion equations. After expressing the mathematical theory of the blow-up by Kaplan and giving a general approach, it was actively studied by researchers.

Various methods are used to investigate blow-up phenomena (see [5, 14]). These methods are generally used to obtain the upper bounds of the blow-up time. If blow-up occurs, little paper were studied to obtain lower and upper bounds on blow-up time. It is important to obtain lower and upper bounds for the process being modeled because of its explosive nature. In [18], a first order differential inequality technique was used to obtain a lower bound on the blow-up in finite time of a semilinear parabolic problem under homogeneous Dirichlet condition by authors (see [17] for homogeneous Neumann boundary condition). In [19] and [20], the authors extended these results to more general nonlinear parabolic problems. In these papers, the authors studied the blow-up time of solutions the equations such that $u_t = \mathrm{div}(\rho|\nabla u|^2)\mathrm{grad}u + f(u)$ and obtained lower bounds by using a differential inequality technique for blow-up time under suitable conditions.

In our this paper, we deal with the following parabolic problem involving generalized source $p(\cdot)$:

$$\begin{cases} u_t = \Delta u + f(x,u), & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, & t > 0, \end{cases} \tag{1}$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$ and the source term as following

$$f(x,u) = \beta u^{p(x)} \text{ or } f(u) = \beta \int_\Omega u^{q(y)}(y,t)dy,$$

and $\beta > 0$ is a parameter. Let $p, q: \Omega \to (1, +\infty)$ generalized sources functions satisfy

$$1 < p^- \leq p(x) \leq p^+ < +\infty \text{ a.e. } x \in \Omega, \tag{2}$$

and

$$1 < q^- \leq q(x) \leq q^+ < +\infty \text{ a.e. } x \in \Omega. \tag{3}$$

We denote $L^h(\Omega)$ usual Lebesgue spaces with the norm $\|\cdot\|_{L^h(\Omega)} := \|\cdot\|_h$ $(1 \leq h \leq \infty)$.

Problem (1) is an expanded version of the problems in the above-mentioned studies. These type problems describes the density of some biological species or the diffusion of concentration of some Newtonian fuids through porous medium in many biological species theories and physical phenomena (see [6, 9]). Under certain conditions on certain ranges of exponents and the initial data, the existence, uniqueness, blow-up and other qualitative properties of solutions for parabolic equations with constant and variable nonlinearity were studied by many authors (see [1, 2, 3, 4, 7, 8, 11, 13, 16, 19] and references therein).

In [21], the author studied upper bound for the blow-up in finite time with initial data which is sufficiently large for positive solutions of parabolic and hyperbolic problems with local and nonlocal type reaction terms involving a variable exponent for following problem:

$$\begin{cases} u_t = \Delta u + f(u), & (x,t) \in \Omega \times [0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T), \end{cases} \tag{4}$$

where the source term is of the form

$$f(x,u) = a(x)u^{p(x)} \text{ or } f(u) = a(x)\int_\Omega u^{q(y)}(y,t)dy,$$

and functions $p, q\colon \Omega \to (1,\infty)$ and the continuous function $a\colon \Omega \to \mathbb{R}$:

$$1 < p^- \leq p(x) \leq p^+ < +\infty, \tag{5}$$

$$1 < q^- \leq q(x) \leq q^+ < +\infty, \tag{6}$$

$$0 < c_a \leq a(x) \leq C_a < +\infty. \tag{7}$$

The author stated the following Theorem and proved the existence of initial data such that the corresponding solutions blow-up at a finite time.

Theorem 1 ***(Theorem 1.1 in*** **[21]*).*** *Assume that* $\Omega \in \mathbb{R}^N$ *is a bounded smooth domain and* $u > 0$ *is a solution of equation* (4), *with* p, q *and* a *satisfying conditions* (5)-(7). *Then, for a sufficiently large initial datum* $u_0(x)$, *there exists a finite time* $T_f > 0$ *such that*

$$\sup_{0 \le t \le T_f} \|u(\cdot,t)\|_\infty = +\infty.$$

The paper is organized as follows. In Section 2, we remember some necessary definitions, introduce some notations and give important lemmas which will be used to prove our main theorems. In Section 3, we obtain the upper bounds for the blow-up time and a criterion for blow-up under some sufficient condition on parameter β. Moreover, we obtain the global solution under the suitable condition on parameter β.

2 Preliminaries

Now, we shall introduce some notation, basic definitions and important lemmas which will be needed in the course of this paper.

We define the function

$$\eta(t) = \int_\Omega u\varphi_1 dx. \tag{8}$$

Let $\lambda_1 > 0$ be the first eigenvalue and $\varphi_1(x) > 0$ be the corresponding (smallest) eigenfunction of the Laplacian in Ω with zero Dirichlet boundary conditions (membrane problem):

$$\{\Delta\varphi + \lambda\varphi = 0, \qquad x \in \Omega, \qquad \varphi|_{\partial\Omega} = 0, \tag{9}$$

and

$$\int_\Omega \varphi_1 dx = 1.$$

Definition 2. *We say that the solution $u(x,t)$ blows up in a finite time if there exists an instant $T_f < +\infty$ such that*

$$\|u(\cdot,t)\|_\infty \to \infty \; as \; t \to T_f.$$

We can easily see that the finite time blow-up happens if there exists a $T_f < +\infty$ such that $\eta(T_f) = +\infty$. Indeed:

$$\eta(t) = \int_\Omega u\varphi_1 dx \le \|u(\cdot,t)\|_\infty \int_\Omega \varphi_1 dx = \|u(\cdot,t)\|_\infty \to \infty \text{ as } t \to T_f.$$

Thanks to this observation, blow-up of the solution $u(x,t)$ can be characterized in terms of the function $\eta(t)$.

We will investigate the upper bound for blow-up time of blow-up solution to problem (1) (see [12]) by using eigenvalue method of Kaplan. This method leads to an ordinary differential inequality of first order which blow-up in finite time.

Lemma 3. *Assume that $\Psi(t)$ is a solution of*

$$\Psi'(t) \geq c\Psi^k(t), \qquad \Psi(0) > 0,$$

where $k > 1$, and $c > 0$. Then, $\Psi(t)$ cannot be globally defined, and

$$\Psi(t) \geq \left(\Psi^{1-k}(0) - \frac{k-1}{c}t\right)^{-\frac{1}{k-1}}.$$

By using Lemma 3, an upper bound can be obtained for the blow-up time.

3 Main Results and Proofs

In this paper we deal with the blow-up for positive solutions of parabolic problem involving generalized sources with local and nonlocal type reaction terms. Based on a modified differential inequality technique, we prove the upper bounds for the blow-up time and a criterion for blow-up under some sufficient condition on parameter β. Moreover, we obtain the global solution under the suitable condition on parameter β.

Our main results are the following theorems:

Theorem 4. *Assume that Ω is a bounded smooth domain in $\mathbb{R}^N$and $u > 0$ is a solution of the problem (1), and function p satisfying condition (2) and*

$$f(x,u) = \beta u^{p(x)}$$

with $\beta > 0$. If

$$\beta > \frac{\lambda_1 \eta(0)}{(\eta^{p^-}(0) - 1)_+},$$

where $\lambda_1 > 0$ is the first eigenvalue of the problem (9), then the problem (1) has no global solutions in finite time $T_f > 0$. We have

$$\int_{\eta(0)}^{+\infty} \frac{d\tau}{\beta\tau^{p^-} - \lambda_1\tau - \beta} \geq T_f,$$

where

$$\eta(0) = \int_{\Omega} u_0\varphi_1 dx.$$

Theorem 5. *Assume that* Ω *is a bounded smooth domain in* $\mathbb{R}^N$*and* $u > 0$ *is a solution of the problem* (1), *and function q satisfying condition* (3) *and*

$$f(u) = \beta \int_{\Omega} u^{q(y)} (y, t) dy$$

with $\beta > 0$. *If*

$$\beta > \frac{\lambda_1\eta(0)\|\varphi_1\|_\infty}{(\eta^{q^-}(0) - 1)_+},$$

where $\lambda_1 > 0$ *is the first eigenvalue, and* $\varphi_1 > 0$ *in* Ω *is the first eigenfunction of the problem* (9), *then the problem* (1) *has no global solutions in finite time* $T_f^0 > 0$. *We have*

$$\int_{\eta(0)}^{+\infty} \frac{\|\varphi_1\|_\infty d\tau}{\beta\tau^{q^-} - \lambda_1\|\varphi_1\|_\infty\tau - \beta} \geq T_f^0.$$

We consider the problem (1) but now ask that p satisfying condition $0 < p^- \leq p^+ \leq 1$. In this case, we show that the solution remains bounded for all time when a restriction is imposed on the constant β.

Theorem 6. *Assume that* Ω *is a bounded smooth domain in* $\mathbb{R}^N$*and* $u > 0$ *is a solution of the problem* (1). *We define the function*

$$\eta(t) = \int_{\Omega} u^2\, dx, \tag{10}$$

with $\|u_0\|_2^2 = \int_\Omega u_0^2\, dx$. *Let*

$$f(x, u) = \beta u^{p(x)}$$

with $\beta > 0$. *If* $0 < p^- \leq p(x) \leq p^+ \leq 1$ *and* β *satisfies the following condition*

$$0 < \beta \leq \frac{\left(2\lambda_1\|u_0\|_2^2 - \frac{1-p^-}{2}\left(|\Omega|^{\frac{2}{1-p^-}} + |\Omega|^{\frac{2}{1-p^+}}\right)\right)_+}{(p^+ + p^- + 2)\|u_0\|_2^2},$$

where $\lambda_1 > 0$ *is the first eigenvalue of the problem* (9), *then solution* u *of the problem* (1) *is bounded.*

Proof of Theorem 4. Taking the scalar product in $L^2(\Omega)$ with φ_1 of both parts of the equation (1) and integrating the resulting expression in t, we obtain the equality

$$\int_\Omega u\varphi_1 dx - b_0 = -\lambda_1 \int_0^t \eta(\xi) d\xi + \beta \int_0^t \int_\Omega u^{p(x)} \varphi_1 dx d\xi, \tag{11}$$

where

$$b_0 = (u_0, \varphi_1) > 0.$$

Next since $\varphi_1(x) > 0$ and $1 < p^- \leq p(x) \leq p^+, \forall x \in \Omega$, we derive that

$$\begin{aligned} \int_\Omega u^{p(x)} \varphi_1 dx &\geq \int_{\Omega \cap \{x: u > 1\}} u^{p^-} \varphi_1 dx \\ &= \int_\Omega u^{p^-} \varphi_1 dx - \int_{\Omega \cap \{x: u \leq 1\}} u^{p^-} \varphi_1 dx \\ &\geq \int_\Omega u^{p^-} \varphi_1 dx - \int_\Omega \varphi_1 dx \\ &= \int_\Omega u^{p^-} \varphi_1 dx - 1. \end{aligned} \tag{12}$$

By (11) and (12), we get

$$\eta(t) - b_0 \geq \beta \int_0^t \int_\Omega (u^{p^-} \varphi_1 - 1) dx d\xi - \lambda_1 \int_0^t \eta(\xi) d\xi. \tag{13}$$

Furthermore, taking into account the fact that $p^- > 1$ by using Hölder's inequality, in (8), we obtain

$$\eta(t) = \int_\Omega u\varphi_1 dx = \int_\Omega u\varphi_1^{\frac{1}{p^-}} \varphi_1^{1-\frac{1}{p^-}} dx \le \left(\int_\Omega u^{p^-}\varphi_1 dx\right)^{\frac{1}{p^-}} \left(\int_\Omega \varphi_1 dx\right)^{\frac{p^- -1}{p^-}}$$

$$= \left(\int_\Omega u^{p^-}\varphi_1 dx\right)^{\frac{1}{p^-}}, \qquad (14)$$

and the fact that $\int_\Omega \varphi_1 dx = 1$. By (11), (13) and (14), we can write

$$\eta(t) - b_0 \ge \beta \int_0^t (\eta^{p^-}(\xi) - 1) d\xi - \lambda_1 \int_0^t \eta(\xi) d\xi. \qquad (15)$$

From (15), we obtain

$$\eta'(t) \ge \beta(\eta^{p^-}(t) - 1) - \lambda_1 \eta(t),\ t > 0 \qquad (16)$$

with

$$\beta > \frac{\lambda_1 \eta(0)}{(\eta^{p^-}(0) - 1)_+}. \qquad (17)$$

From (16), we get

$$\eta'(t) \ge f(\eta(t)),\ t > 0, \qquad (18)$$

where $f(s) = \beta s^{p^-} - s - \beta$, and now the result follows from Lemma 3 with (17). Namely, the function η^{p^-} is monotone increasing for all $t \ge 0$ because of $\eta'(t) > 0$, then the solution of the problem (1) blows up in finite time. Therefore the solution of the boundary value problem is unbounded. Moreover, dividing the both parts of (18) by $f(s)$ and integrating, we get

$$I(\eta) = \int_{\eta(0)}^{\eta(t)} \frac{ds}{f(s)} \ge t. \qquad (19)$$

Because of the $I(s)$ is convergent at $s = +\infty$ with $p^- > 1$, the inequality (19) is possible only if there exists T_f such that $\overline{\lim_{t\to T_f}} \eta(t) \to \infty$. Therefore u cannot exist globally. Thus the Theorem 4 is proved.

Proof of Theorem 5. Let consider the function f as

$$f(u) = \beta \int_{\Omega} u^{q(y)}(y,t)dy$$

with $\beta > 0$. In much the same way, we obtain

$$\eta(t) - b_0 = -\lambda_1 \int_0^t \eta(\xi)d\xi + \beta \int_0^t \int_{\Omega} \left(\int_{\Omega} u^{q(y)}(y,\xi)dy \right) \varphi_1(x)dxd\xi, \quad (20)$$

where

$$b_0 = (u_0, \varphi_1) > 0.$$

Similarly relation (12), we have

$$\int_{\Omega} u^{q(x)} \varphi_1 dx \geq \int_{\Omega} u^{q^-} \varphi_1 dx - 1, \forall y \in \Omega. \quad (21)$$

By (20) and (21), we derive

$$\eta(t) - b_0 \geq -\lambda_1 \int_0^t \eta(\xi)d\xi + \beta \int_0^t \int_{\Omega} (u^{q^-}(y,\xi) - 1)dy \int_{\Omega} \varphi_1(x)\, dxd\xi$$

$$\geq -\lambda_1 \int_0^t \eta(\xi)d\xi + \frac{\beta}{\|\varphi_1\|_\infty} \int_0^t \int_{\Omega} (u^{q^-}(y,\xi) - 1)\varphi_1(y)dyd\xi$$

$$\geq -\lambda_1 \int_0^t \eta(\xi)d\xi + \frac{\beta}{\|\varphi_1\|_\infty} \int_0^t (\eta^{q^-}(\xi) - 1)d\xi, \quad (22)$$

since

$$\eta^{q^-}(t) \leq \int_{\Omega} u^{q^-} \varphi_1 dy. \quad (23)$$

Then from (22) and (23), we have

$$\eta(t) - b_0 \geq -\lambda_1 \int_0^t \eta(\xi)d\xi + \frac{\beta}{\|\varphi_1\|_\infty} \int_0^t (\eta^{q^-}(\xi) - 1)d\xi.$$

Similarly (18), we have

$$\eta'(t) \geq \frac{\beta}{\|\varphi_1\|_\infty} \eta^{q^-}(t) - \lambda_1 \eta(t) - \frac{\beta}{\|\varphi_1\|_\infty} \equiv g(\eta(t)), \qquad t > 0. \quad (24)$$

Since $q^- > 1$ and

$$\beta > \frac{\lambda_1 \eta(0)\|\varphi_1\|_\infty}{(\eta^{q^-}(0) - 1)_+}, \tag{25}$$

and by Lemma 3 with (25), the solution of the problem (1) blows up in finite time. Moreover, dividing the both parts of (24) by $g(s)$ and integrating, we see that

$$J(\eta) = \int_{\eta(0)}^{\eta(t)} \frac{ds}{g(s)} \geq t. \tag{26}$$

Because of the $J(s)$ is convergent at $s = +\infty$ with $q^- > 1$, the inequality (26) is possible only if there exists T_f^0 such that $\overline{\lim_{t \to T_f^0}} \eta(t) \to \infty$. Therefore u cannot exist globally. The proof of Theorem 5 is completed.

Proof of Theorem 6. By using (10) and first equation of problem (1), we compute

$$\begin{aligned}
\mu'(t) &= 2\int_\Omega u u_t dx \\
&= 2\int_\Omega u\big(\Delta u + \beta u^{p(x)}\big)dx = -2\int_\Omega |\nabla u|^2 dx + 2\beta \int_\Omega u^{p(x)+1} dx \\
&\leq -2\int_\Omega |\nabla u|^2 dx + 2\beta\left(\int_\Omega u^{p^-+1} dx + \int_\Omega u^{p^++1} dx\right) \\
&\leq -2\|\nabla u\|_2^2 + 2\beta\left(\int_\Omega u^{p^-+1} dx + \int_\Omega u^{p^++1} dx\right),
\end{aligned} \tag{27}$$

since

$$u^{p(x)+1} \leq u^{p^++1} + u^{p^-+1}, \forall x \in \Omega.$$

By using Young's inequality for the last integrals with $0 < p^-, p^+ < 1$, we obtain

$$\int_\Omega u^{p^-+1} dx \leq \frac{p^- + 1}{2}\int_\Omega u^2 dx + \frac{1 - p^-}{2}|\Omega|^{\frac{2}{1-p^-}}, \tag{28}$$

and

$$\int_\Omega u^{p^+ +1} dx \le \frac{p^+ + 1}{2} \int_\Omega u^2 dx + \frac{1 - p^+}{2} |\Omega|^{\frac{2}{1-p^+}}. \tag{29}$$

Let u be unbounded at some time T. From the Rayleigh principle

$$\lambda_1 \int_\Omega v^2 dx \le \int_\Omega |\nabla v|^2 dx,$$

where $\lambda_1 > 0$ is the first eigenvalue of the problem (9). From (27), (28) and (29), we have

$$\eta'(t) \le [(p^+ + p^- + 2)\beta - 2\lambda_1] \int_\Omega u^2 dx + \frac{1 - p^-}{2} |\Omega|^{\frac{2}{1-p^-}} + \frac{1 - p^+}{2} |\Omega|^{\frac{2}{1-p^+}}.$$

If we restrict $\beta > 0$ such that

$$\beta \le \frac{2\lambda_1 \|u_0\|_2^2 - \frac{1-p^-}{2}\left(|\Omega|^{\frac{2}{1-p^-}} + |\Omega|^{\frac{2}{1-p^+}}\right)}{(p^+ + p^- + 2)\|u_0\|_2^2} \le \frac{\left(2\lambda_1 \|u_0\|_2^2 - \frac{1-p^-}{2}\left(|\Omega|^{\frac{2}{1-p^-}} + |\Omega|^{\frac{2}{1-p^+}}\right)\right)_+}{(p^+ + p^- + 2)\|u_0\|_2^2}, \tag{30}$$

we easily get $\mu'(t) \le 0$.

Moreover it must be noticed that the blow-up time is T (supposes to exist), but $\mu'(t) \le 0$ holds for every time t, which implies that u is bounded. This is a contradiction. We can obtain that there is no time T such that u is unbounded. Therefore u is bounded for every time t. Thus, Theorem 6 is proved.

Remark 7. *When* $p(x) \equiv p = 1, \forall x \in \Omega,$ $i.e. p^+ = p^- = p = 1$ *in the condition* (30)*, we have*

$$\beta \le \frac{\lambda_1}{2}.$$

Therefore, we can obtain that there is no time T such that the solution u of the problem (1) *is unbounded for any* u_0.

References

[1] S. Antontsev, S. Shmarev, Blow-up of solutions to parabolic equations with nonstandard growth conditions, J. Comput. Appl. Math., Vol. 234, No. 9, pp. 2633-2645 (2010).

[2] S. Antontsev, S. Shmarev, On the blow-up of solutions to anisotropic parabolic equations with variable nonlinearity, Proc. Steklov Inst. Math., Vol. 270, No. 1, pp. 27-42 (2010).

[3] R. Ayazoglu (Mashiyev), E. Akkoyunlu, Extinction properties of solutions for a parabolic equation with a parametric variable exponent nonlinearity, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, Vol. 42, No. 1, pp.1-16 (2022).

[4] K. Baghaei, M.B. Ghaemi, M. Hesaaraki, Lower bounds for the blow-up time in a semilinear parabolic problem involving a variable source, Appl. Math. Lett., Vol. 27, pp. 49-52 (2014).

[5] C. Bandle and H. Brunner, Blow-up in diffusion equations: a survey, J. Comput. Appl. Math., Vol. 97, pp. 3-22 (1998).

[6] J. Bebernes, A. Bressan, Thermal behavior for a concerned reactive gas, J. Differ. Equ., Vol. 44, No. 1, pp. 118-133 (1982).

[7] M. M. Bokalo, O. M. Buhrii, R. A. Mashiyev, Unique solvablity of initial boundary-value problems for anisotropic elliptic-parabolic equations with variable exponents of nanlinearity, J. Nonl. Evol. Equ. Appl., Vol. 2013, No. 6, pp. 67-87 (2014).

[8] O. M. Buhrii, R. A. Mashiyev, Uniqueness of solutions of the parabolic variational inequality with variable exponent of nonlinearity, Nonlinear Anal., Vol. 70, No. 6, pp. 2325-2331 (2009).

[9] J. Furter, M. Grinfield, Local vs. nonlocal interactions in populations dynamics, J. Math. Biol., Vol. 27, No. 1, pp. 65-80 (1989).

[10] V. A. Galaktionov and J. L. Vázquez, The problem of blow-up in nonlinear parabolic equations, Discrete Cont. Dyn. Sys., Vol. 8, pp. 399-433 (2002).

[11] M. A. Herrero, J. J. L. Velazquez, Blow-up behaviour of one-dimensional semilinear parabolic equations, Ann. Inst. H. Poincare Anal. Non Lineaire, Vol. 10, No. 2, pp. 131-189 (1993).

[12] S. Kaplan, On the growth of the solutions of quasilinear parabolic equations, Comm. Pure Appl. Math., Vol. 16, No. 3, 305-330 (1963).

[13] O. Kholyavka, O. M. Buhrii, M. Bokalo, R. Ayazoglu (Mashiyev), Initial boundary-value problem for third order equations of Kirchhoff type with variable exponents of nonlinearity, Advances in Math. Sciences and Appl., Vol. 23, No. 2, pp. 509-528 (2013).

[14] H. A. Levine, Nonexistence of global weak solution to some properly and improperly posed problems of mathematical physics: the method of unbounded Fourier coefficients , Math. Ann., Vol. 214, pp. 205-220 (1975).

[15] H. A. Levine, The role of critical exponents in blow-up theorems, SIAM Reviw, Vol. 32, pp. 262-288 (1990).

[16] R. A. Mashiyev, O. M. Buhrii, Existence of solutions of the parabolic variational inequality with variable exponent of nonlinearity, J. Math. Anal. Appl., Vol. 377, No. 2, pp. 450-463 (2011).

[17] L. E. Payne and P. W. Schaefer, Lower bounds for blow-up time in parabolic problems under Neumann conditions, Appl. Anal., Vol. 85, pp. 1301-1311 (2006).

[18] L. E. Payne and P. W. Schaefer, Lower bounds for blow-up time in parabolic problems under Dirichlet conditions, J. Math. Anal. Appl., Vol. 328, pp. 1196-1205 (2007).

[19] L. E. Payne, G. A. Philippin, P. W. Schaefer, Blow-up phenomena for some nonlinear parabolic problems, Nonlinear Anal., Vol. 69, No. 10, pp. 3495-3502 (2008).

[20] L. E. Payne, G. A. Philippin, and P. W. Schaefer, Bounds for blow-up time in nonlinear parabolic problems, J. Math. Anal. Appl., Vol. 338, pp. 438-447 (2008).

[21] J. P. Pinasco, Blow-up for parabolic and hyperbolic problems with variable exponents, Nonlinear Anal., Vol. 71, No. 3-4, pp. 1094-1099 (2009).

[22] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, Blow-up in Quasilinear Parabolic Equations, Walter de Gruyter, Berlin/New York (1995).

[23] B. Straughan, Explosive Instabilities in Mechanics, Springer, Berlin (1998)