Bölüm 2

A Generalization of Szász Type Operators Involving Generating Function of Negative Order Genocchi Polynomials

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Abstract

In this chapter, using the generating function for obtaining the Szász type operators, we define an extension that includes Genocchi type polynomials. We investigate convergence properties such as modulus of continuity, Korovkin-Bohman theorem, Lipschitz class by using some of important definitions, equalitiy and inequality.

Introduction

The Genocchi polynomials are one of the important special polynomial families. Genocchi polynomials are named after its creator, Angelo Genocchi (1817-1899). The generating function of Genocchi polynomials is defined to be is defined to be

$$\left(\frac{2t}{e^t+1}\right)e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},|t|<\pi.$$

If we put $x=0$, The Genocchi numbers's generating function is described as follows:

$$\left(\frac{2t}{e^t+1}\right)=\sum_{n=0}^{\infty}G_n(0)\frac{t^n}{n!}.$$

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The generating functions have a lot of bridges between mathematics and other applied sciences such as Combinatorics, Computer Aided Geometric Design (CAGD) and Machine learning. The generating functions are designed to help us effectively convert problems involving sequences into problems involving functions. Many researchers have studied with Genocchi polynomials and numbers by means of their Generating functions. Dere and Simsek derived properties of the Genocchi polynomials and constructed relations between Genocchi polynomials of higher order and other special polynomials and numbers of higher order such as Euler polynomials and Stirling numbers [1]. Kilar and Simsek studied some relations and formulas including the Genocchi polynomials of negative order and the other special polynomials and numbers [2]. Kim obtained some relations and formulas for q- analogue of Genocchi polynomials and The modified poly-Genocchi polynomials and numbers obtained from the modified degenerate poly-exponential function were studied by Kim *et al.*, who also came up with conclusions and identities regarding those polynomials and numbers as well as some other unique numbers and polynomials [3]. Srivastava *et al.* proposed a new form Euler type polynomials by aid of their generating function and presented n analogue abstraction of the closely-related Genocchi type polynomials [4]. A. F. Horadam defined negative order Genocchi polynomials by means of their generating functions as follows:

$$\left(\frac{1+e^t}{2t}\right)^k e^{tx} = \sum_{n=-k}^{\infty} G_n^{-k}(x)\frac{t^n}{n!}, \tag{1}$$

where $k \in \mathbb{N}$ [5].

Nowadays, the generating functions of special polynomial sequences are one of the important tools to construct positive linear operators. Brenke form polynomials were used by Varma *et al.* to introduce novel Szász form operators and study the convergence characteristics of these operators [6]. Using Brenke form polynomials, Taşdelen *et al*. created a Kantorovich variant of Szász form operators and found convergence characteristics like a Voronovskaya-type finding [7]. Chlodowsky modified Szász form operators were provided by Mursaleen *et al.* along with Boas-Buck type polynomials and derived convergence features like the continuity modulus [8]. Prakash *et al.* suggested a new type sequence of operators that includes Apostol-Genocchi polynomials Via the use of their generating functions [9]. Menekşe Yılmaz presented a new Szász type operator by using Apostol-Genocchi polynomials of order α and studied approximation properties of these operators [10]. Gezer and Menekşe Yilmaz studied convergence properties of a generalizations of Kantorovich form operators which including Charlier polynomialsand gave Voronovskaya type theorem by using an asymptotic formula [11]. See the following references for more information [12-17].

By the motivation above studies and the Eq. (1), we define our operator at the following definition:

Definition 1: Let $G_n^{-1}(x)$ be -1 order Genocchi polynomials. The operator is defined to be

$$\mathcal{G}_n^*(f,x) = \frac{2}{e+1} e^{-nx} \sum_{k=0}^{\infty} \frac{G_k^{-1}(x)}{k!} f\left(\frac{k}{n}\right). \tag{2}$$

Main Results

In this section, we investigate the convergence properties of $\mathcal{G}_n^*(f,x)$. To show the uniformly convergence of $\mathcal{G}_n^*(f,x)$, we give the moment functions of operator at the following lemma:

Lemma 1: To all $x \in [0, \infty)$, Eq. (2) provides at the aforementioned equalities:

$$\mathcal{G}_n^*(1, x) = 1 \tag{3}$$

$$\mathcal{G}_n^*(s, x) = x + \frac{2e+1}{n(e+1)} \tag{4}$$

$$\mathcal{G}_n^*(s^2, x) = x^2 + \frac{5e+3}{n(e+1)}x + \frac{5e+1}{n^2(e+1)} \tag{5}$$

Proof: By using derivatives the generating functions of the -1 order Genocchi polynomials, we have:

$$\sum_{k=0}^{\infty} G_k^{-1}(x)\frac{t^k}{k!} = \left(\frac{1+e^t}{2t}\right)^1 e^{tx},$$

$$\sum_{k=1}^{\infty} G_k^{-1}(x)k\frac{t^{k-1}}{k!} = \frac{1}{2}e^{tx}(xt + e^t(xt + t + 1) + 1),$$

and

$$\sum_{k=1}^{\infty} G_k^{-1}(x)k(k-1)\frac{t^{k-2}}{k!} = \frac{1}{2}e^{tx}(x(xt + 2) + e^t(x + 1)(xt + t + 2))$$

$t = 1$ and x is replacing by nx throughout the proof. Using the definition of $\mathcal{G}_n^*(f, x)$, it's simple to see

$$\mathcal{G}_n^*(1, x) = \frac{2}{e+1}e^{-nx}\sum_{k=0}^{\infty}\frac{G_k^{-1}(nx)}{k!} = \frac{2}{e+1}e^{-nx}\frac{e+1}{2}e^{nx} = 1.$$

Let $f = s$.

$$\mathcal{G}_n^*(s, x) = \frac{2}{e+1}e^{-nx}\sum_{k=0}^{\infty}\frac{G_k^{-1}(nx)}{k!}\left(\frac{k}{n}\right) = \frac{2}{e+1}e^{-nx}\frac{1}{2n}e^{nx}(nx + e(nx + 2) + 1)$$

$$= \frac{1}{(e+1)n}[nx(e + 1) + 2e + 1]$$

$$= x + \frac{2e+1}{n(e+1)}$$

Let $f = s^2$.

$$\mathcal{G}_n^*(s^2,x) = \frac{2e^{-nx}}{e+1}\left[\frac{1}{2}e^{nx}(nx+e(nx+2)+1)+\frac{1}{2}e^{nx}(nx(nx+2)+e(nx+1)(nx+3)\right]$$

$$= \frac{2e^{-nx}}{e+1}\frac{e^{nx}}{2n^2}[(nx+e(nx+2)+1)+(nx(nx+2)+e(nx+1)(nx+3)))]$$

$$= \frac{1}{(e+1)n^2}[nx(e+1)+(2e+1)+(nx)^2(e+1)+6nx+3e]$$

$$= x^2+\frac{5e+3}{n(e+1)}x+\frac{5e+1}{n^2(e+1)}.$$

Now, using the moment functions at lemma 1 and Korovkin's theorem in [18], we demonstrate the uniform convergence of $\mathcal{G}_n^*(f,x)$ as follows:

Theorem 2: We assume that $f \in [0,\infty] = C[0,\infty] \cap E$. Then

$$\lim_{n\to\infty} \|\mathcal{G}_n^* f - f\| = 0,$$

where $C[0,\infty]$ is continuously functions space, $E = \left\{f: x \in [0,\infty), \lim_{x\to-\infty} \frac{f(x)}{1+x^2}\ exist\right\}$ and $f(s) = s^i$ for $i = 0,1,2$.

Proof: Take into consideration the well-known Lemma 1 and the Korovkin's first theorem. Then, we obtain

$$\lim_{n\to\infty} \mathcal{G}_n^*\left(s^i,x\right) = x^i, for\ i = 0,1,2.$$

The operator $\mathcal{G}_n^*(f,x)$ satisfies converge consistently in every compact subset of $[0,\infty]$. The property (vii) of theorem 4.1.4 is used to accomplish the desired result in [18].

By using linearity propert of $\mathcal{G}_n^*(f,x)$, we give the 2^{nd} order central moment at the following equation:

$$\mathcal{G}_n^*((s-x)^2,x) = \mathcal{G}_n^*(s^2,x) - 2x\mathcal{G}_n^*(s,x) + x^2\mathcal{G}_n^*(1,x) \tag{6}$$

Putting Eq. (3) –(5) along the right side of Eq. (6), we have

$$\mathcal{G}_n^*((s-x)^2,x) = x^2 + \frac{x}{n}\frac{5e+3}{e+1} + \frac{5e+1}{n^2(e+1)} - 2x^2 - \frac{x(4e+2)}{n(e+1)} + x^2 \tag{7}$$

After arranging Eq. (7), we get at the following lemma:

Lemma 3: To all $x \in [0,\infty)$, Eq. (2) provides the aforementioned equalities:

$$\mathcal{G}_n^*((s-x)^2,x) = \frac{x}{n} + \frac{5e+1}{n^2(e+1)}. \tag{8}$$

Now, we give estimate the order of $\mathcal{G}_n^*(f,x)$ by using modulus of continuity. The modulus of continuity is defined to be

$$\omega(f,\delta) = sup|f(x) - f(y)|, \tag{9}$$

where $f \in [0,\infty]$ and $\delta > 0$. One of the important properties of modulus of continuity is

$$|f(x) - f(y)| \leq \omega(f,\delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{10}$$

Theorem 4: We assume that $f \in [0,\infty] = C[0,\infty] \cap E$. The operator $\mathcal{G}_n^*(f,x)$ provides at the following inequality:

$$|\mathcal{G}_n^*(f,x) - f(x)| \leq 2\omega(f,\sqrt{\delta_n}),$$

where $\delta_n = \sqrt{\mathcal{G}_n^*((s-x)^2,x)}$.

Proof: Using the notion of the continuity modulus and Eq. (10), we have

$$|\mathcal{G}_n^*(f,x) - f(x)| \leq \mathcal{G}_n^*(|f(s) - f(x)|;x) \leq \omega(f,\delta)\left(\frac{\mathcal{G}_n^*(|x-y|,x)}{\delta} + 1\right) \tag{11}$$

If we apply the Cauchy-Schwartz inequality to right hand side of Eq. (11), we reach

$$|\mathcal{G}_n^*(f,x)-f(x)|\leq\omega(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\mathcal{G}_n^*((s-x)^2,x)}\right).$$

By picking $\delta\coloneqq\delta_n=\sqrt{\mathcal{G}_n^*((s-x)^2,x)}$, the proof is completed.

To examine estimation of the convergence rate of $\mathcal{G}_n^*(f,x)$, we construct the theorem containing the Lipschitz class of order α which is another method. The definition of Lipschitz class of order α is given as

$$Lip_M(\alpha)\coloneqq\{f\epsilon C_B[0,\infty):|f(t)-f(x)|\leq M|t-x|^\alpha;t,x\epsilon[0,\infty)\},\tag{12}$$

where $f\in C_B[0,\infty)$ and $\alpha\in(0,1],M>0$.

Theorem 5: For $f\in Lip_M(\alpha)$ and $x\epsilon[0,\infty)$, we give the following inequality for $\mathcal{G}_n^*(f,x)$

$$|\mathcal{G}_n^*(f,x)-f(x)|\leq M\vartheta_n^\alpha(x).\tag{13}$$

Proof: Because the $\mathcal{G}_n^*(f,x)$ is monotonic, we give

$$|\mathcal{G}_n^*(f,x)-f(x)|\leq M\mathcal{G}_n^*(f,x)(|s-x|^\alpha;x).\tag{14}$$

By applying the Hölder inequality to Eq.(14), we obtain at the following inequality:

$$|\mathcal{G}_n^*(f,x)-f(x)|\leq M\left(\mathcal{G}_n^*((s-x)^2,x)\right)^{\frac{\alpha}{2}}.\tag{15}$$

Therefore, the desired result is obtained.

Conclusion

In this chapter, we defined a generalizations of Szász form operators involving (-1) order Genocchi polynomials by means of their generating functions. However, we investigated uniformly convergence property of $\mathcal{G}_n^*(f,x)$ by using moment functions and Korovkin's theorem. By obtaining second order central moment functions for $\mathcal{G}_n^*(f,x)$, we presented an estimate to rate of convergence using the continuity modulus. And also, we gave some theorems to estimate the convergence rate of our operator such as Lipschitz class order α.

Because special polynomials and their generating functions have many applied areas, future works can be a bridge between approximation theory and special polynomials such as Bernoulli polynomials, Euler polynomials, Genocchi polynomials and Fubini type polynomials. By using q-generalizations of generating functions of special polynomials, the new type generalizations of positive linear operators can be defined and investigated their convergence properties. However, Many different generalizations of linear positive operators can construct by using generating functions method such as Kantrovich Szász form operators. And also, the Voronovskaya and Voronovskaya-Grüss theorems for new operators can be studied.

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References

[1] Dere, R., & Simsek, Y. (2011). Genocchi polynomials associated with the Umbral algebra. *Applied Mathematics and Computation*, *218*(3), 756-761.

[2] Kilar, N. (2022). Formulas for special numbers and polynomials derived from functional equations of their generating functions. *Turkish Journal of Mathematics*, *46*(6), 2450-2465.

[3] Kim, T. (2007). A note on the-Genocchi numbers and polynomials. *Journal of Inequalities and Applications*, *2007*, 1-8.

[4] Srivastava, H. M., Garg, M., & Choudhary, S. (2011). Some new families of generalized Euler and Genocchi polynomials. *Taiwanese Journal of Mathematics*, *15*(1), 283-305.

[5] Horadam, A. F. (1992). Negative order Genocchi polynomials. Fibonacci Q, 30, 21-34.

[6] Varma, S., Sucu, S., & İçöz, G. (2012). Generalization of Szász operators involving Brenke type polynomials. *Computers & Mathematics with Applications*, *64*(2), 121-127.

[7] Taşdelen, F., Aktaş, R., & Altın, A. (2012, January). A Kantorovich type of Szász operators including Brenke-type polynomials. In *Abstract and Applied Analysis* (Vol. 2012). Hindawi.

[8] Mursaleen, M., Al-Abied, A. A. H., & ACU, A. (2018). Approximation by Chlodowsky type of Szász operators based on Boas--Buck-type polynomials. *Turkish Journal of Mathematics*, *42*(5), 2243-2259.

[9] Prakash, C., Verma, D. K., & Deo, N. (2021). Approximation by a new sequence of operators involving Apostol-Genocchi polynomials. *Mathematica Slovaca*, *71*(5), 1179-1188.

[10] Meneks Yilmaz, M. (2022). Approximation by Szász Type Operators Involving Apostol-Genocchi Polynomials. *CMES-COMPUTER MODELING IN ENGINEERING & SCIENCES*, *130*(1), 287-297.

[11] Gezer, K. & Menekşe Yılmaz, M. (). Approximation Properties of a Class of Kantorovich Type Operators Associated with the Charlier Polynomials . Yüzüncü Yıl Üniversitesi Fen Bilimleri Enstitüsü Dergisi , DOI: 10.53433/yyufbed.1187512

[12] İspir, N., Aral, A., & Doğru, O. (2008). On Kantorovich process of a sequence of the generalized linear positive operators. *Numerical Functional Analysis and Optimization*, *29*(5-6), 574-589.

[13] Jung, H. S., Deo, N., & Dhamija, M. (2014). Pointwise approximation by Bernstein type operators in mobile interval. *Applied Mathematics and Computation*, *244*, 683-694.

[14] Kajla, A., & Agrawal, P. N. (2016). Szász-Kantorovich type operators based on Charlier polynomials. *Kyungpook Mathematical Journal*, *56*(3), 877-897.

[15] Kajla, A., & Agrawal, P. N. (2015). Approximation properties of Szász type operators based on Charlier polynomials. *Turkish Journal of Mathematics*, *39*(6), 990-1003.

[16] Varma, S., & Taşdelen, F. (2012). Szász type operators involving Charlier polynomials. *Mathematical and Computer Modelling*, *56*(5-6), 118-122.

[17] Verma, D. K., & Gupta, V. (2016). Approximation by a New Sequence of Operators Involving Charlier Polynomials with a Certain Parameter.

In *Modern Mathematical Methods and High Performance Computing in Science and Technology: M3HPCST, Ghaziabad, India, December 2015* (pp. 25-34). Springer Singapore.

[18] Paltanea, R. (2004). *Approximation theory using positive linear operators*. Springer Science & Business Media.