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Optimality Conditions of a Hyperbolic Beam Equation based on Mindlin's Gradient Elasticity Theory

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Abstract

In this study, optimality conditions of a beam model based on Mindlin's gradient elasticity theory is studied. The beam system depends on the external excitation function, non-homogeneous boundary conditions and some mixed integral constraints including ineqality/equality on the control function and state variable. Before obtaining the optimality conditions of the system, energy integral method is employed for proving the uniqueness of the solution of the beam system. Controllability properties of the system is also discussed. Adjoint system corresponding to beam system is derived with suitable terminal conditions for achieving the maximum principle. *The beauty of the present paper is that the necessary and sufficient optimality conditions of a hyperbolic beam equation based on Mindlin's gradient elasticity theory are firstly derived in this paper in the form of a maximum principle.* In order to show the confirmation of the obtained theoretical results, a real mechanical problem is illustrated and results are presented in the table and graphical forms.

1 Introduction

The contributions of classical continuum theories, including nonlinear or linear plasticity and elasticity, to science and engineering to improve the human life quality by modeling solid and structures are deniable. These continuum theories which are also named Euler-Bernoulli or Timoshenko models were introduced in 1750s for explaining the conservation laws of solid and structures in macro-scales. After 1920s, especially later than detailed usage of advanced optical and electron microscopes, the dimensions of structures and systems in engineering and material science are scaled down to micro and nano-domains. The elasticity properties and characteristic behaviors of materials in the micro-nano domains was also tried to explained by means of classical continuum models at the beginning. But observations made by advanced electron and optical microscopes show that classical continuum theories are not able to explain the characteristic behaviors and elasticity properties of micro or nano-scaled solid and structures due to lack of an internal length scale parameters, characteristic of the underlying nano or micro structured materials, from the constitutive equations. In order to overcome this difficulties, several studies and theories are introduced in the papers [1]-[11]. In [8], Mindlin introduced and developed gradient elasticity theory which is comprehension of linear elasticity theory which contains higher-order terms to taking into account for structural effects in micro-size or couple stress effects in materials. Mindlin improved his theory by including new terms in the expressions of potential and kinetic energy and introducing intrinsic micro-structural parameter without however providing explicit expressions that correlate micro-struc-ture with macro-structure [13]. According to Mindlin's this theory, the energy in the strain is subject to the elastic strain and gradients of the elastic strain. Due to these gradients, constitutive equations includes additional coefficients with the dimension of

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a length which are called gradient coefficients. For sake, see, [1]-[13]. Since then, many researchers studying strain gradient elastic theories, reproduced either from lattice models or homogenization approaches, have presented in the literature. Although gracious, none of them derives as a whole the equation of motion as well as the on-classical or classical boundary conditions seeming in Mindlin theory, in terms of the taken into account lattice or continuum unit cell. Moreover, no continuum or lattice models that affirm the second gradient elastic theory of Mindlin presented in the literature. In [13], authors introduce a model equation of motion, appearing in Mindlin theory, confirming the Mindlin second gradient elastic theory. On the other hand, in order to determine the necessary and sufficient optimality conditions for these kind vibrating systems, Maximum principle is introduced by L. S. Pontryagin 1960s as a necessary condition for optimal control problems representing in way of ordinary differential equations[14]. In [15], Egorov also shows that maximum principle is also necessary requirement for some class optimal control problems modeled by partial differential equations. In [16, 17], Barnes and Lee proved that maximum principle is sufficient requirement for control problems under some convexity assumptions on the constraints functions. Russell and Komkov studied for obtaining the necessary and sufficient conditions of similar vibrating systems including quadratic cost functionals[18, 19].In [20], Active control of an improved Boussinesq system is achieved via maximum principle. In [21], Necessary and sufficient conditions for a vibrating Euler-Bernoulli beam system, including control functions more than one, is achieved in the form of maximum principle. In [22], necessary and sufficient conditions of a distributed parameter system is derived. The original contribution of the present paper to literature is that the necessary and sufficient optimality conditions of a beam system, satisfying Mindlin gradient elasticity theory, is firstly derived in the form of maximum principle in this paper.

Specifically, in the light of [21] and [22] in present study, necessary and sufficient optimality conditions of a beam model satisfying Mindlin's gradient elasticity theory is studied. The beam system under consideration depends on the external excitation function caused to undesirable vibration in the system, non-homogeneous boundary conditions showing thermal or magnetically effects and some mixed integral constraints including in-eqality/equality on the control function and state variable. Before obtaining the optimality conditions of the system, by employing the energy integral method, the uniqueness of the solution to the beam system is proved. Controllability properties of the system is also discussed via observability. Adjoint system corresponding to beam system is introduced with suitable terminal conditions for achieving the maximum principle. In order to indicate the confirmation of the obtained theoretical results, a numerical example is given and results are presented in the table and graphical forms. By observing the table and graphics, it is that introduced conditions in the form of maximum principle are necessary and sufficient for optimality. Consider the following beam system defined in [13];

$$\frac{\partial^2 \nu}{\partial x^2} + \frac{\ell^2}{12}\frac{\partial^4 \nu}{\partial x^4} + \frac{\ell^4}{36}\frac{\partial^6 \nu}{\partial x^6} = \frac{1}{c^2}\left(\frac{\partial^2 \nu}{\partial t^2} - \frac{\ell^2}{3}\frac{\rho'}{\rho}\frac{\partial^4 \nu}{\partial t^2 \partial x^2}\right) + f(t,x) + \mathcal{C}(t,x) \tag{1}$$

where $\nu(t,x)$ is the transversal displacement at $(t,x) \in \Omega = \{(t,x) : t \in [0,t_f], x \in [0,\ell]\}$, x is space variable, ℓ is the length of the beam, t is time variable, t_f is the final time of control duration, $c^2 = \frac{E}{\rho}$ in which ρ is mass density of the beam and E is modulus elasticity, $\rho' \equiv \rho$ is the the density of the micro-structural cells, f is external excitation function, $\mathcal{C}(t,x)$ is control function, $\mathcal{C}(t,x) = \mathcal{C}(t,x)$ or generally $\mathcal{C}(t,x) = C(t)D(x)$ in which $C(t)$ is the control force function and $D(t)$ is a function presenting the distribution of the control force, $\mathcal{C}(t,x) \in \mathfrak{C}_{ad}$ in which $\mathfrak{C}_{ad}$ is the set of admissible control functions and defined by $\mathfrak{C}_{ad} = \{\mathcal{C}(t,x) | \mathcal{C}(t,x) \in L^2(\Omega), \quad |\mathcal{C}(t,x)| \leq \mathfrak{m} < \infty\}$, $\mathfrak{m}$ is a constant. Eq. (1) is subject to the following initial conditions

$$\nu(0,x) = \nu_0(x), \quad \frac{\partial \nu}{\partial t}(0,x) = \nu_1(x), \tag{2}$$

$$\text{in which} \quad \nu_0(x) \in H^1(0,\ell), \quad H^1(0,\ell) = \{u_0(x) \in L^2(0,\ell) : \frac{\partial u_0(x)}{\partial x} \in L^2(0,\ell)\}, \quad \nu_1(x) \in L^2(0,\ell),$$

and $L^2(\Omega)$ presents the Hilbert space of real-valued square-integrable functions defined in the domain Ω with following inner product and norm given in the sense of Lebesque;

$$\| \phi \|^2 = <\phi,\phi>, \quad <\phi,\varphi>_\Omega = \int_\Omega \phi\varphi d\Omega.$$

Eq.(1) is subject to following non-homogeneous boundary conditions

$$\nu(t,0)=\zeta_1(t),\quad \frac{\partial^2\nu}{\partial x^2}(t,0)=\zeta_2(t),\quad \frac{\partial^4\nu}{\partial x^4}(t,0)=\zeta_3(t),\quad \nu(t,\ell)=\zeta_4(t),\quad \frac{\partial^2\nu}{\partial x^2}(t,\ell)=\zeta_5(t),\quad \frac{\partial^4\nu}{\partial x^4}(t,\ell)=\zeta_6(t), \tag{3}$$

or

$$\frac{\partial\nu}{\partial x}(t,0)=\xi_1(t),\quad \frac{\partial^3\nu}{\partial x^3}(t,0)=\xi_2(t),\quad \frac{\partial^5\nu}{\partial x^5}(t,0)=\xi_3(t),\quad \frac{\partial\nu}{\partial x}(t,\ell)=\xi_4(t),\quad \frac{\partial^3\nu}{\partial x^3}(t,\ell)=\xi_5(t),\quad \frac{\partial^5\nu}{\partial x^5}(t,\ell)=\xi_6(t). \tag{4}$$

Let us make the following assumptions on the system;

(A1) $\frac{\partial^i\nu}{\partial t^i},\frac{\partial^j\nu}{\partial x^j},\frac{\partial^{m+n}\nu}{\partial x^m\partial t^n},\in L^2(\bar{\Omega})$, $\bar{\Omega}$ is closure of Ω, $i=0,1,2$ $j=0,1,...,6$, $m,n=0,1,2$,

(A2) $\zeta_i(t),\xi_i(t)\in L^2(\Omega)$, $i=1,...,6$.

Then, the system addressed by Eqs.(1)-(4) has a solution[27].

Lemma 1. *The system called by Eqs.*(1) *-(4) has a unique solution.*

Proof. Let us that ν_1 and ν_2 are two solutions to the system under the same conditions. Then the difference $u=\nu_1-\nu_2$ satisfies the following homogeneous initial conditions

$$u(x,t)=0,\quad u_t(x,t)=0 \quad \text{at} \quad t=0 \tag{5}$$

and boundary conditions

$$u(t,0)=0,\quad \frac{\partial^2u}{\partial x^2}(t,0)=0,\quad \frac{\partial^4u}{\partial x^4}(t,0)=0,\quad u(t,\ell)=0,\quad \frac{\partial^2u}{\partial x^2}(t,\ell)=0,\quad \frac{\partial^4u}{\partial x^4}(t,\ell)=0, \tag{6a}$$

or

$$\frac{\partial u}{\partial x}(t,0)=0,\quad \frac{\partial^3u}{\partial x^3}(t,0)=0,\quad \frac{\partial^5u}{\partial x^5}(t,0)=0,\frac{\partial u}{\partial x}(t,\ell)=0,\quad \frac{\partial^3u}{\partial x^3}(t,\ell)=0,\quad \frac{\partial^5u}{\partial x^5}(t,\ell)=0, \tag{6b}$$

and equation of motion becomes as follows;

$$u_{xx}+\frac{\ell^2}{12}u_{xxxx}+\frac{\ell^4}{36}u_{xxxxxx}-\frac{1}{c^2}u_{tt}+\frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}u_{ttxx}=0 \tag{7}$$

Let us show that u is identically equal to zero. Then, introduce the following energy integral;

$$E(t)=\frac{1}{2}\int_0^\ell\left\{\frac{\partial^2}{\partial x^2}(u^2)+\frac{\ell^2}{12}\frac{\partial^4}{\partial x^4}(u^2)+\frac{\ell^4}{36}\frac{\partial^6}{\partial x^6}(u^2)-\frac{1}{c^2}(u_t^2)+\frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}\frac{\partial^2}{\partial x^2}(u_t^2)\right\}dx \tag{8}$$

and show that $E(t)$ is independent of t. Differentiating $E(t)$ with respect to t, it is easy to see following equality;

$$\frac{dE(t)}{dt}=\int_0^\ell\left\{\frac{\partial^2}{\partial x^2}(uu_t)+\frac{\ell^2}{12}\ell^2\frac{\partial^4}{\partial x^4}(uu_t)+\frac{\ell^4}{36}\frac{\partial^6}{\partial x^6}(uu_t)-\frac{1}{c^2}(u_tu_{tt})+\frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}\frac{\partial^2}{\partial x^2}(u_tu_{tt})\right\}dx \tag{9}$$

Integrating by parts and using boundary conditions indicated by Eq.(6), Eq.(9) becomes

$$\frac{dE(t)}{dt}=\int_0^\ell\left\{u_{xx}+\frac{\ell^2}{12}u_{xxxx}+\frac{\ell^4}{36}u_{xxxxxx}-\frac{1}{c^2}u_{tt}+\frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}u_{ttxx}\right\}u_tdx. \tag{10}$$

Due to Eq.(7), following equality is observed

$$\frac{dE(t)}{dt}=0,\quad \text{that is} \quad E(t)=\text{constant}.$$

Regarding the conditions defined by Eq.(5), one obtains;

$$E(0)=\frac{1}{2}\int_0^\ell\left\{\frac{\partial^2}{\partial x^2}(u^2)+\frac{\ell^2}{12}\frac{\partial^4}{\partial x^4}(u^2)+\frac{\ell^4}{36}\frac{\partial^6}{\partial x^6}(u^2)-\frac{1}{c^2}(u_t^2)+\frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}\frac{\partial^2}{\partial x^2}(u_t^2)\right\}\Bigg|_{t=0}dx=0.$$

Hence, it becomes obviously that $u(x,t)$ is zero, identically and $u=\nu_1-\nu_2=0\Rightarrow\nu_1=\nu_2$. Namely, the system under consideration has a unique solution. □

By considering Lemma 1, it is concluded that for saving the uniqueness of the solution $\nu(x,t)$, corresponding control function $\mathcal{C}(t,x)$ has to be unique. In this case, it is said that the system under consideration has a unique solution $\nu(x,t)$ and a unique control function $\mathcal{C}(t,x)$. Then, system introduced Eqs.(1)-(4) is referred as observable. Hilbert Uniqueness method indicates that observable is equivalent to controllable. As a conclusion, Eqs.(1)-(4) is controllable [25, 26].

2 Optimal Control Problem

The main goal of the optimal control problem is to determine optimal control function $\mathcal{C}^{\circ}(t,x) \in \mathfrak{C}_{ad}$ minimizing the performance index functional at a given terminal time t_f. The performance index functional of the system, is consisted of a sum of modified energy of the beam and control effort used up in control duration, introduced as follows;

$$\begin{aligned}\mathcal{J}_0(f(t,x)) = &\int_0^{\ell} [\mathcal{G}_1(x,\nu(t_f,x)) + \mathcal{G}_2(x,\nu_t(t_f,x))]dx \\ &+ \int_0^{t_f}\int_0^{\ell} \mathcal{G}_0(t,x,\nu(t,x),\mathcal{C}(t,x))dtdx.\end{aligned} \tag{11}$$

Admissible control function $\mathcal{C}(t,x)$ subject to the Eqs.(1)-(2) and the following constraints

$$\int_0^{\ell} h_2(x,\nu_t(t_f,x)dx + \int_0^{t_f}\int_0^{\ell} \mathcal{G}_{-2}(t,x,\nu(t,x),\mathcal{C}(t,x))dtdx = c_{-2}, \tag{12a}$$

$$\int_0^{\ell} h_1(x,\nu(t_f,x)dx + \int_0^{t_f}\int_0^{\ell} \mathcal{G}_{-1}(t,x,\nu(t,x),\mathcal{C}(t,x))dxdt = c_{-1}, \tag{12b}$$

$$\int_0^{t_f}\int_0^{\ell} \mathcal{G}_i(t,x,\nu(t,x),\mathcal{C}(t,x))dxdt \le c_i, \quad 1 \le i \le m, \tag{12c}$$

$$\int_0^{t_f}\int_0^{\ell} \mathcal{G}_i(t,x,\nu(t,x),\mathcal{C}(t,x))dxdt = c_i, \quad m < i \le M \tag{12d}$$

in which, for $i = -2,-1,1,...,M$, $h_1, h_2, \mathcal{G}_0, \mathcal{G}_i$ are continuous functions of their all parameters. Also, $h_1, \mathcal{G}_0, \mathcal{G}_i$ for $i = -2,-1,1,...,M$ are the functions having continuous derivation respect to ν. Also, $h_2, \mathcal{G}_2$ are functions having continuous derivation respect to ν_t. Suppose that $\mathcal{C}^{\circ}(t,x)$ is optimal control function with corresponding to optimal displacement ν°. By assuming $(t_1,x_1),...,(t_P,x_P)$ are P arbitrary points in the open region Ω and $\mathcal{C}_j(t,x), \quad j = 1,...,P$ are P arbitrary subfunctions of admissible control function $\mathcal{C} \in \mathfrak{C}_{ad}$. Also, let us assume that $x_1 \le x_2 \le ... \le x_P$. Let $\varsigma > 0$ be for $x_i + P\varsigma < x_j$ if $x_i < x_j, \quad x_P + P\varsigma < \ell$ and $t_i + \varsigma < t_f$ for each $0 \le i \le P$. Let $\varepsilon_1,...,\varepsilon_P$ be real parameters satisfying $0 \le \varepsilon_j \le \varsigma^2$. Let $X_1 = x_1$ and $X_j = x_j + \sqrt{\varepsilon_1} + ... + \sqrt{\varepsilon_{j-1}}$ be for $1 < j \le P$. Hence, the intervals $X_j \le x \le X_j + \sqrt{\varepsilon_j}$ and the rectangles $R_j : [t_j, t_j + \sqrt{\varepsilon_j}] \times [X_j, X_j + \sqrt{\varepsilon_j}]$ do not have any intersection for $1 \le j \le P$, respectively. ε denotes the vector $(\varepsilon_1,...,\varepsilon_P) \in \mathbb{R}^P$, $\mathbb{R}^P$ is a space in the manner of $P-$ dimensional Euclidean, and $\varepsilon = |\varepsilon| = \varepsilon_1 + ... + \varepsilon_P$. Control $\mathcal{C}_\varepsilon(t,x) \in \bar{\Omega}$ is defined by

$$\mathcal{C}_\varepsilon(t,x) = \begin{cases} \mathcal{C}^{\circ}(t,x) & \text{if} \quad (t,x) \notin \bigcup_{j=1}^{P} R_j, \\ \mathcal{C}(t,x) & \text{if} \quad (t,x) \in R_j, \quad j = 1,...,P \end{cases} \tag{13}$$

3 Adjoint System and Optimality Conditions

Necessary requirement for optimality is obtained by means of maximum principle. Assuming by some convexity conditions on the constraints, maximum principle seems sufficient requirement for optimality. In order to constructing the maximum principle, an adjoint variable $v(t,x) \in \Omega^*$, in which Ω^* is the dual to Ω having same norm and inner product like in Ω, along the adjoint operator is defined. The $v(t,x)$ satisfies the following adjoint equation;

$$\frac{\partial^2 v}{\partial x^2}+\frac{\ell^2}{12}\frac{\partial^4 v}{\partial x^4}+\frac{\ell^4}{36}\frac{\partial^6 v}{\partial x^6}=\frac{1}{c^2}\left(\frac{\partial^2 v}{\partial t^2}-\frac{\ell^2}{3}\frac{\rho^{'}}{\rho}\frac{\partial^4 v}{\partial t^2\partial x^2}\right)+\sum_{i=-2}^{M}\lambda_i\frac{\partial\mathcal{G}_i}{\partial\nu}(t,x,\nu^\circ,\mathcal{C}^\circ(t,x)), \tag{14}$$

where $\lambda_i \le 0$ and Eq.(14) is subject to the following homogeneous boundary conditions

$$v(t,0)=0,\quad \frac{\partial^2 v}{\partial x^2}(t,0)=0,\quad \frac{\partial^4 v}{\partial x^4}(t,0)=0,\quad v(t,\ell)=0,\quad \frac{\partial^2 v}{\partial x^2}(t,\ell)=0,\quad \frac{\partial^4 v}{\partial x^4}(t,\ell)=0, \tag{15a}$$

or

$$\frac{\partial v}{\partial x}(t,0)=0,\quad \frac{\partial^3 v}{\partial x^3}(t,0)=0,\quad \frac{\partial^5 v}{\partial x^5}(t,0)=0,\quad \frac{\partial v}{\partial x}(t,\ell)=0,\quad \frac{\partial^3 v}{\partial x^3}(t,\ell)=0,\quad \frac{\partial^5 v}{\partial x^5}(t,\ell)=0, \tag{15b}$$

and the terminal conditions

$$\frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}v_{txx}(t,x)-\frac{1}{c^2}v_t(t,x)=\lambda_{-1}\frac{\partial h_1}{\partial\nu}(x,\nu(t,x))+\lambda_0\frac{\partial\mathcal{G}_1}{\partial\nu}(x,\nu(t,x))\quad\text{at}\quad t=t_f \tag{16a}$$

$$v(t,x)=-\lambda_0\frac{\partial\mathcal{G}_2}{\partial\nu_t}(x,\nu_t(t,x))-\lambda_{-2}\frac{\partial h_2}{\partial\nu_t}(x,\nu_t(t,x))\quad\text{at}\quad t=t_f. \tag{16b}$$

Existence and uniqueness of the solution corresponding to Eqs.(14)-(16) can be shown similarly to Eqs.(1)-(4).

Lemma 2. *Let v and $\Delta\nu(t,x)=\nu(t,x)-\nu^\circ(t,x)$ be two functions which are defined in $L^2(\Omega)$. Also, let us assume that v and $\Delta\nu(t,x)$ satisfy Eqs.(14)-(16) and Eqs.(1)-(4), respectively. Then,*

$$\int_0^{t_f}\int_0^{\ell}\Big\{v[\frac{\partial^2\Delta\nu}{\partial x^2}+\frac{\ell^2}{12}\frac{\partial^4\Delta\nu}{\partial x^4}+\frac{\ell^4}{36}\frac{\partial^6\Delta\nu}{\partial x^6}-\frac{1}{c^2}\left(\frac{\partial^2\Delta\nu}{\partial t^2}-\frac{\ell^2}{3}\frac{\rho^{'}}{\rho}\frac{\partial^4\Delta\nu}{\partial t^2\partial x^2}\right)]$$
$$-\Delta\nu[\frac{\partial^2 v}{\partial x^2}+\frac{\ell^2}{12}\frac{\partial^4 v}{\partial x^4}+\frac{\ell^4}{36}\frac{\partial^6 v}{\partial x^6}-\frac{1}{c^2}\left(\frac{\partial^2 v}{\partial t^2}-\frac{\ell^2}{3}\frac{\rho^{'}}{\rho}\frac{\partial^4 v}{\partial t^2\partial x^2}\right)]\Big\}dxdt$$
$$=\int_0^{\ell}\Big\{v(t_f,x)\frac{\ell^2}{3}\frac{\rho^{'}}{\rho}\frac{1}{c^2}\Delta\nu_{txx}(t_f,x)-v_t(t_f,x)\frac{\ell^2}{3}\frac{\rho^{'}}{\rho}\frac{1}{c^2}\Delta\nu_{xx}(t_f,x)-\frac{1}{c^2}\Delta\nu_t(t_f,x)v(t_f,x)+\frac{1}{c^2}v_t(t_f,x)\Delta\nu(t_f,x)\Big\}dx \tag{17}$$

Proof. After applying the integration by parts to Eq.(17), it is easy to see the conclusion of lemma 2. □

Definition 1. *v is the solution of the system Eqs.(14)-(16) for the arbitrary constants $\lambda_{-2},\lambda_{-1},\lambda_0,...,\lambda_M$. Let ν° and ν be the response functions corresponding to optimal control functions $\mathcal{C}^\circ\in\mathfrak{C}_{ad}$, and for a fixed $\mathcal{C}\in\mathfrak{C}_{ad}$, u be the any one of the following functions:*

$$u(t,x)=\mathcal{G}_i(t,x,\nu,\mathcal{C}(t,x)),$$
$$u(t,x)=\mathcal{G}_i(t,x,\nu^\circ,\mathcal{C}^\circ(t,x)),$$
$$u(t,x)=v\mathcal{C}^\circ,\quad \mathcal{C}^\circ\in\mathfrak{C}_{ad},$$
$$u(t,x)=v\mathcal{C},\quad \mathcal{C}\in\mathfrak{C}_{ad}\quad \textit{is fixed}\quad .$$

A point $(\bar t,\bar x)$ is named regular point for $\mathcal{C}\in\mathfrak{C}_{ad}$ if it meets the following equality for any sufficiently small $\varepsilon>0$

$$\int_{\bar t}^{\bar t+\sqrt{\varepsilon}}\int_{\bar x}^{\bar x+\sqrt{\varepsilon}}u(t,x)dxdt=\varepsilon u(\bar t,\bar x)+o(\varepsilon).$$

By considering [23], it is reveals that all points of $[0,t_f]\times[0,\ell]$ are regular for each $\mathcal{C}\in\mathfrak{C}_{ad}$.

Let $\mathcal{J}$ and Z indicate the vector valued functional $(\mathcal{J}_{-2},\mathcal{J}_{-1},\mathcal{J}_0,...,\mathcal{J}_M)$ and the set

$$Z=\{\mathcal{J}(\mathcal{C}):\mathcal{C}\in\mathfrak{C}_{ad}\}\subset\mathbb{R}^{M+3}.$$

Definition 2. *If a surface exists in the following form*

$$\mathcal{J}_\varepsilon = \mathcal{J}(\mathcal{C}^\circ) + \sum_{j=1}^{P} d_j \varepsilon_j + o(\varepsilon)$$

in Z for sufficiently small ε_j and $d_1, ..., d_P$ is any finite collection of vectors from D, then set D is called as a derived set of the set Z at $\mathcal{J}(\mathcal{C}^\circ)$ [24].

Lemma 3. *Assume that the points (t_i, x_i) are regular points in Ω for $i = 1, 2, ..., P$. Let us introduce the $\bar{\mathcal{J}}(\mathcal{C})$ for any $\mathcal{C} \in \mathfrak{C}_{ad}$ as follows;*

$$\bar{\mathcal{J}}(\mathcal{C}) = \int_0^\ell \Big\{ \lambda_0 [\mathcal{G}_1(x, \nu(t_f, x)) + \mathcal{G}_2(x, \nu_t(t_f, x))]$$

$$+ [\lambda_{-2} h_2(x, \nu_t(t_f, x)) + \lambda_{-1} h_1(x, \nu(t_f, x))] \Big\} dx$$

$$+ \int_0^{tf} \int_0^\ell \sum_{i=-2}^{M} \big[\lambda_i \mathcal{G}_i(t, x, \nu, \mathcal{C})\big] dt dx$$

If $P = 1$, for $\mathcal{C} \in \mathfrak{C}_{ad}$, there exist constants $\lambda_{-2}, \lambda_{-1}, \lambda_0, ..., \lambda_M$ (not all zero) such that

$$\lambda_0 \le 0, \quad \lambda_i \le 0 \quad (0 \le i \le m), \quad \lim_{\varepsilon \to 0+} \frac{\bar{\mathcal{J}}(\mathcal{C}_\varepsilon) - \bar{\mathcal{J}}(\mathcal{C}^\circ)}{\varepsilon} \le 0$$

where $\mathcal{C}^\circ$'s and $\mathcal{C}_\varepsilon$'s are functions introduced in Eq.(13).

Proof. Define the functionals $\mathcal{J}_{-2}, \mathcal{J}_{-1}, \mathcal{J}_1, ..., \mathcal{J}_M$ on the class of admissible controls by

$$\mathcal{J}_{-2}(\mathcal{C}) = \int_0^\ell h_2(x, \nu_t(t_f, x)) dx + \int_0^{tf} \int_0^\ell G_{-2}(t, x, \nu, \mathcal{C}) dx dt,$$

$$\mathcal{J}_{-1}(\mathcal{C}) = \int_0^\ell h_1(x, \nu_t(t_f, x)) dx + \int_0^{tf} \int_0^\ell G_{-1}(t, x, \nu, \mathcal{C}) dx dt,$$

$$\mathcal{J}_i(\mathcal{C}) = \int_0^{tf} \int_0^\ell G_i(t, x, \nu, \mathcal{C}) dx dt, \quad i = 1, ..., M.$$

For employing Lagrange multipliers, we need to construct a derived set D for the set Z at $\mathcal{J}(\mathcal{C}^\circ)$ [24]. Now let us define the functions v_j for $j = -2, -1, 1, ..., M$ which are supplying the following conditions:

$$\frac{\partial^2 v_j}{\partial x^2} + \frac{\ell^2}{12} \frac{\partial^4 v_j}{\partial x^4} + \frac{\ell^4}{36} \frac{\partial^6 v_j}{\partial x^6} = \frac{1}{c^2} \left(\frac{\partial^2 v_j}{\partial t^2} - \frac{\ell^2}{3} \frac{\rho'}{\rho} \frac{\partial^4 v_j}{\partial t^2 \partial x^2} \right) + \lambda_j \frac{\partial \mathcal{G}_j}{\partial \nu}(t, x, \nu^\circ, \mathcal{C}^\circ(t, x)) \tag{18}$$

where $\lambda_i \le 0$, $0 \le t \le t_f$, $0 \le x \le \ell$ and Eq.(18) is subject to the following boundary conditions

$$v_j(t, 0) = 0, \quad \frac{\partial^2 v_j}{\partial x^2}(t, 0) = 0, \quad \frac{\partial^4 v_j}{\partial x^4}(t, 0) = 0, \quad v_j(t, \ell) = 0, \quad \frac{\partial^2 v_j}{\partial x^2}(t, \ell) = 0, \quad \frac{\partial^4 v_j}{\partial x^4}(t, \ell) = 0 \tag{19a}$$

or

$$\frac{\partial v_j}{\partial x}(t, 0) = 0, \quad \frac{\partial^3 v_j}{\partial x^3}(t, 0) = 0,, \quad \frac{\partial^5 v_j}{\partial x^5}(t, 0) = 0, \quad \frac{\partial v_j}{\partial x}(t, \ell) = 0, \quad \frac{\partial^3 v_j}{\partial x^3}(t, \ell) = 0, \quad \frac{\partial^5 v_j}{\partial x^5}(t, \ell) = 0, \quad -2 \le j \le M \tag{19b}$$

and the terminal conditions

$$v_j(t_f, x) = 0, \quad \frac{1}{c^2} \frac{\ell^2}{3} \frac{\rho'}{\rho} \frac{\partial^3}{\partial t \partial x^2} v_j(t_f, x) - \frac{1}{c^2} \frac{\partial}{\partial t} v_j(t_f, x) = 0 \quad \text{for} \quad i = 1, .., M. \tag{20a}$$

$$v_{-2}(t_f, x) = -\lambda_{-2} \frac{\partial h_2}{\partial \nu_t}(x, \nu_t(t_f, x)), \quad \frac{1}{c^2} \frac{\ell^2}{3} \frac{\rho'}{\rho} \frac{\partial^3}{\partial t \partial x^2} v_{-2}(t_f, x) - \frac{1}{c^2} \frac{\partial}{\partial t} v_{-2}(t_f, x) = 0, \tag{20b}$$

$$v_{-1}(t_f,x)=0,\quad \frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}\frac{\partial^3}{\partial t\partial x^2}v_{-1}(t_f,x)-\frac{1}{c^2}\frac{\partial}{\partial t}v_{-1}(t_f,x)=\lambda_{-1}\frac{\partial h_1}{\partial \nu}(x,\nu(t_f,x)) \tag{20c}$$

$$v_0(t_f,x)=-\lambda_0\frac{\partial \mathcal{G}_2}{\partial \nu_t}(x,\nu_t(t_f,x)), \tag{20d}$$

$$\frac{1}{c^2}\frac{\ell^2}{3}\frac{\rho'}{\rho}\frac{\partial^3}{\partial t\partial x^2}v_0(t_f,x)-\frac{1}{c^2}\frac{\partial}{\partial t}v_0(t_f,x)=\lambda_0\frac{\partial \mathcal{G}_1}{\partial \nu}(x,\nu(t_f,x)).$$

For each point $(t,x)\in(0,t_f)\times(0,\ell), i=-2,-1,0,...,M,$ $d^i(t,x,\bar{\mathcal{C}})$ is defined as follows

$$d^i(t,x,\bar{\mathcal{C}})=v_i(t,x)(\bar{\mathcal{C}}-\mathcal{C}^\circ)+\mathcal{G}_i(t,x,\nu^\circ(t,x),\bar{\mathcal{C}})-\mathcal{G}_i(t,x,\nu^\circ(t,x),\mathcal{C}^\circ). \tag{21}$$

Now we shall show that the set

$$D=\{d|d=(d^{-2}(t,x,\bar{\mathcal{C}}),...,d^M(t,x,\bar{\mathcal{C}})),(t,x)\quad \text{a regular point of}\quad \mathcal{C}^\circ,\bar{\mathcal{C}}\in\mathfrak{C}_{ad}\}$$

is derived set for Z at $\mathcal{J}(\mathcal{C}^\circ)$. Let $d_1,d_2,...,d_P$ be an arbitrary finite collection of vectors from D. We must show that there exist points $\mathcal{J}_\varepsilon\in Z$ depending continuously on the vector parameter $\varepsilon=(\varepsilon_1,...,\varepsilon_P)$ for all sufficiently small positives values of ε such that

$$\mathcal{J}_\varepsilon=\mathcal{J}(\mathcal{C}^\circ)+\sum_{j=1}^{P}d_j\varepsilon_j+o(\varepsilon).$$

Since $d_j\in D,\quad j=1,...,P,\quad$ there exist $(t_1,x_1),...,(t_P,x_P)$ regularity points of $\mathcal{C}^\circ$ and subfunctions $\mathcal{C}_1,...,\mathcal{C}_P\in\mathfrak{C}_{ad}$ such that

$$d_j=(d^{-2}(t_j,x_j,\mathcal{C}_j),...,d^M(t_j,x_j,\mathcal{C}_j)),j=1,...,P.$$

We shall show that $\mathcal{J}_\varepsilon$ can be defined by $\mathcal{J}_\varepsilon=\mathcal{J}(\mathcal{C}_\varepsilon)$ where $\mathcal{C}_\varepsilon$ is the admissible control defined in Eq.(13). Then, for $i=1,...,M$

$$\begin{aligned}\mathcal{J}_i(\mathcal{C}_\varepsilon)-\mathcal{J}_i(\mathcal{C}^\circ)&=\int_0^{t_f}\int_0^{\ell}[\mathcal{G}_i(t,x,\nu_\varepsilon(t,x),\mathcal{C}_\varepsilon(t,x))-\mathcal{G}_i(t,x,\nu^\circ(t,x),\mathcal{C}^\circ(t,x))]dtdx\\&=\int_0^{t_f}\int_0^{\ell}[\mathcal{G}_i(t,x,\nu_\varepsilon(t,x),\mathcal{C}_\varepsilon(t,x))-\mathcal{G}_i(t,x,\nu^\circ(t,x),\mathcal{C}_\varepsilon(t,x))]dtdx\\&+\int_0^{t_f}\int_0^{\ell}[\mathcal{G}_i(t,x,\nu^\circ(t,x),\mathcal{C}_\varepsilon(t,x))-\mathcal{G}_i(t,x,\nu^\circ(t,x),\mathcal{C}^\circ(t,x))]dtdx\end{aligned} \tag{22}$$

$$\begin{aligned}=\int_0^{t_f}\int_0^{\ell}\frac{\partial \mathcal{G}_i}{\partial \nu}(t,x,\nu^\circ,\mathcal{C}^\circ(t,x))\Delta\nu(x,t)dtdx+\sum_{j=1}^{P}\varepsilon_j[\mathcal{G}_i(t_j,x_j,\nu^\circ(t_j,x_j),\mathcal{C}_j)\\-\mathcal{G}_i(t_j,x_j,\nu^\circ(t_j,x_j),\mathcal{C}_j^\circ(t_j,x_j))]+\sum_{j=1}^{P}o(\varepsilon_j).\end{aligned} \tag{23}$$

For obtaining Eq.(23), we employ that $\mathcal{C}^\circ$ is regular in Ω. After employing following equality in Eq.(17)

$$\mathfrak{M}v_i=\frac{\partial \mathcal{G}_i}{\partial \nu}(t,x,\nu^\circ(t,x),\mathcal{C}^\circ),\quad i=1,...,M,\quad k=1,...,N$$

it is observed that

$$\begin{aligned}\int_0^{\ell}\int_0^{t_f}\Delta\nu(t,x)\mathfrak{M}v_idtdx&=\int_0^{\ell}\int_0^{t_f}v_i(t,x)(\mathcal{C}_\varepsilon(t,x)-\mathcal{C}^\circ(t,x))\\&=\sum_{j=1}^{P}\varepsilon_jv_i(t_j,x_j)(\mathcal{C}_j-\mathcal{C}_j^\circ(t_j,x_j))+o(\varepsilon).\end{aligned}$$

By means of Eq.(21) and Eq.(23), we can write

$$\mathcal{J}_i(\mathcal{C}_\varepsilon) = \mathcal{J}_i(\mathcal{C}^\circ) + \sum_{j=1}^{P} d_j^i \varepsilon_j + o(\varepsilon), \quad i = 1, ..., M, \tag{24}$$

where d_j^i denotes the ith component of d_j. For $i = 0$, we have

$$\begin{aligned}\mathcal{J}_0(\mathcal{C}_\varepsilon) - \mathcal{J}_0(\mathcal{C}^\circ) = &\int_0^\ell [\frac{\partial \mathcal{G}_1}{\partial \nu}(x, \nu^\circ(t_f, x))\Delta\nu(t_f, x) + \frac{\partial \mathcal{G}_2}{\partial \nu_t}(x, \nu_t^\circ(t_f, x))\Delta\nu_t(t_f, x)]dx \\ &+ \sum_{j=1}^{P} \varepsilon_j[\mathcal{G}_0(t_j, x_j, \nu_\varepsilon(t_j, x_j), \mathcal{C}_j) - \mathcal{G}_0(t_j, x_j, \nu_\varepsilon(t_j, x_j), \mathcal{C}_j^\circ)] \\ &+ \int_0^\ell \int_0^{t_f} (\mathcal{M}v_0)\Delta\nu(t, x)dtdx + o(\varepsilon).\end{aligned} \tag{25}$$

in which $\mathcal{J}_0 = \int_0^\ell [\mathcal{G}_1(x, \nu(t_f, x)) + \mathcal{G}_2(x, \nu_t(t_f, x))]dx + \int_0^{t_f} \int_0^\ell \mathcal{G}_0(t, x, \nu, \mathcal{C})dtdx$. Considering Eq.(17) and Eq.(18), it is observed that

$$\begin{aligned}\int_0^{t_f} \int_0^\ell \Delta\nu(t, x)\mathfrak{M}v_0 dtdx = &\int_0^{t_f} \int_0^\ell v_0(t, x)(\mathcal{C}_\varepsilon(t, x) - f^\circ(t, x))dtdx \\ -\int_0^\ell &\left[\frac{\partial \mathcal{G}_1}{\partial \nu}(x, \nu^\circ(t_f, x))\Delta\nu(t_f, x) + \frac{\partial \mathcal{G}_2}{\partial \nu_t}(x, \nu_t(t_f, x))\Delta\nu_t(t_f, x)\right].\end{aligned} \tag{26}$$

If Eq.(26) is substituted into Eq.(25), Eq.(24) is obtained for $i = 0$. For $i = -2, -1$, Eq.(24) can be obtained by using Eqs.(17)-(18). By definition $\mathcal{J}$, following equality is obtained

$$\mathcal{J}(\mathcal{C}_\varepsilon) - \mathcal{J}(\mathcal{C}^\circ) = \sum_{j=1}^{P} d_j \varepsilon_j + o(\varepsilon).$$

If $\mathcal{J}_\varepsilon$ is defined as $\mathcal{J}(\mathcal{C}_\varepsilon)$, it follows that D is a derived set for Z at $\mathcal{J}(\mathcal{C}^\circ)$. So, there exist lagrange multipliers [24] that $\lambda_{-2}, \lambda_{-1}, \lambda_0, ..., \lambda_M$ and $\lambda_i \leq 0$ for $0 \leq i \leq m$ and some $\lambda_i \neq 0$, such that

$$\sum_{i=-2}^{M} \lambda_i d^i \leq 0 \tag{27}$$

for any vector $d = (d^{-2}, d^{-1}, d^0, ..., d^M)$ in D. For attaining conclusion of **Lemma 3**, take into account $P = 1$ and consider

$$\bar{\mathcal{J}} = \sum_{i=-2}^{M} \lambda_i \mathcal{J}_i.$$

By Eq.(24), we have following equality

$$\bar{\mathcal{J}}(\mathcal{C}_\varepsilon) - \bar{\mathcal{J}}(\mathcal{C}^\circ) = \varepsilon \sum_{i=-2}^{M} \lambda_i d^i + o(\varepsilon)$$

for any $d = (d^{-2}, d^{-1}, d^0, ..., d^M)$ in D. Then, we obtain the proof of **Lemma 3** as follows:

$$\lim_{\varepsilon \to 0^+} \frac{\bar{\mathcal{J}}(\mathcal{C}_\varepsilon) - \bar{\mathcal{J}}(\mathcal{C}^\circ)}{\varepsilon} = \sum_{i=-2}^{M} \lambda_i d^i \leq 0.$$

□

Theorem 1. *[Maximum Principle] For the optimal control functions $\mathcal{C}^\circ(t,x) \in \mathfrak{C}_{ad}$, the corresponding optimal state and adjoint variables are let $\nu^\circ(t,x) = \nu(t,x,\mathcal{C}^\circ)$ satisfying Eqs.(1)-(4) and $v^\circ(t,x) = v(t,x,\mathcal{C}^\circ(t,x))$ satisfying Eq.(14), boundary conditions Eq.(15) and terminal conditions Eq.(16), respectively. The maximum principle states that if*

$$\mathcal{H}[t,x,v^\circ,\mathcal{C}^\circ(t,x)] = \max_{\mathcal{C}\in\mathfrak{C}_{ad}} \mathcal{H}[t,x,v,\mathcal{C}(t,x)] \tag{28}$$

where the Hamiltonian is given by

$$\mathcal{H}[t,x,v,\mathcal{C}(t,x)] = v(t,x)\mathcal{C} + \sum_{i=-2}^{M} \lambda_i \mathcal{G}_i(t,x,\nu(t,x),\mathcal{C}) \tag{29}$$

then the performance index Eq.(11) is minimized, i.e.,

$$\mathcal{J}_0[\mathcal{C}^\circ(t,x)] \le \mathcal{J}_0[\mathcal{C}(t,x)] \quad \text{for any} \quad \mathcal{C} \in \mathfrak{C}_{ad}. \tag{30}$$

Proof. By **Lemma 3** and Lagrange multipliers $0 \le i \le m$, $\lambda_i \neq 0$ and $\lambda_{-2}, \lambda_{-1}, \lambda_0, ..., \lambda_M$ independent of (t,x) with $\lambda_i \le 0$ such that

$$\sum_{i=-2}^{M} \lambda_i[v_i(t,x)(\mathcal{C} - \mathcal{C}^\circ(t,x)) + \mathcal{G}_i(t,x,\nu(t,x),\mathcal{C}) - \mathcal{G}_i(t,x,\nu^\circ(t,x),\mathcal{C}^\circ(t,x))] \le 0 \tag{31}$$

for any function $\mathcal{C} \in \mathfrak{C}_{ad}$. Note that the term in Eq.(31)

$$\sum_{i=-2}^{M} \lambda_i[v_i(t,x)\mathcal{C} + \mathcal{G}_i(t,x,\nu(t,x),\mathcal{C})] \tag{32}$$

obtains its maximum value at $\mathcal{C} = \mathcal{C}^\circ(t,x) \in \mathfrak{C}_{ad}$. Take into account the first term in Eq.(32),

$$\sum_{i=-2}^{M} \lambda_i v_i(t,x)\mathcal{C}.$$

If we define $v = \sum_{i=-2}^{M} \lambda_i v_i(t,x)$, we obtain

$$v(t,x)\mathcal{C} + \sum_{i=-2}^{M} \lambda_i \mathcal{G}_i(t,x,\nu(t,x),\mathcal{C}).$$

Hence, **Theorem 1** is proofed. □

Theorem 2. *Take into account the system Eqs.(1)-(2) and Eqs.(11)-(12). Let the functions G_i in the following form*

$$G_i(t,x,\nu,\mathcal{C}) = \mathcal{G}^i(t,x,\nu) + H^i(t,x,\mathcal{C}), \quad i = -2,-1,0,...,M$$

and v satisfying Eqs.(14)-(16) be the nonzero solution of

$$\mathfrak{M}v = \sum_{i=-2}^{M} \lambda_i \frac{\partial G^i(t,x,\nu^\circ(t,x))}{\partial \nu}.$$

Assume presence of admissible control function $\mathcal{C}^\circ$ with the λ_0, λ_i, $i = -2,-1,1,...,M$, that satisfy the maximum principle Eq.(28). Let followings are assumed:

- *a) $\mathcal{G}_1, h_1, \mathcal{G}_3, ..., \mathcal{G}_m$ are convex respect to ν and $\mathcal{G}_2, h_2$ are convex respect to ν_t;*
- *b) $\lambda_0 < 0, \quad \lambda_i \le 0$ for $i = -1,...,m$;.*
- *c) the constraints Eq.(12) are satisfied by $\mathcal{C}^\circ$;*
- *d) If the strict inequality holds in Eq.(12), the corresponding Lagrange multiplier $\lambda_i = 0$;*
- *e) $-\lambda_i G^i, -\lambda_{-1}h_1$ are convex functions of ν and $-\lambda_{-2}h_2$ is convex functions of ν_t for $m < i < M$.*

In presence of above requirements, maximum principle introduced by Eq.(28) is sufficient requirement for $\mathcal{C}^\circ$ to be optimal. Requirement (d) is indicated in [24]. In case $h_1, h_2, G_i, m < i \leq M$ are linear function, requirement (e) is proved.

Proof. If Eq.(12) is satisfied by $\mathcal{C}, \nu$, then by condition (d),

$$\int_0^{t_f}\int_0^{\ell} \lambda_i \Big[G^i(t,x,\nu) - G^i(t,x,\nu^\circ)\Big] dxdt + \int_0^{t_f}\int_0^{\ell} \lambda_i \Big[H^i(t,x,\mathcal{C}) - H^i(t,x,\mathcal{C}^\circ)\Big] dtdx = 0$$

for $\quad i = -2, -1, ..., M$ Then, following inequality can be written;

$$-\lambda_0[\mathcal{J}_0(\mathcal{C}) - \mathcal{J}_0(\mathcal{C}^\circ)] \geq$$

$$-\int_0^{\ell} \lambda_0[\mathcal{G}_2(x,\nu_t(t_f,x)) - \mathcal{G}_2(x,\nu_t^\circ(t_f,x)) + \mathcal{G}_1(x,\nu(t_f,x)) - \mathcal{G}_1(x,\nu^\circ(t_f,x))]dx$$

$$-\int_0^{t_f}\int_0^{\ell} \sum_{i=-2}^{M} \lambda_i \Big\{ G^i(t,x,\nu(t,x)) - G^i(t,x,\nu^\circ(t,x))$$
$$-[H^i(t,x,\mathcal{C}(t,x)) - H^i(t,x,\mathcal{C}^\circ(t,x))]\Big\} dxdt$$

$$-\int_0^{\ell} \lambda_{-1}\Big[h_1(x,\nu(t_f,x)) - h_1(x,\nu^\circ(t_f,x))\Big]dx$$
$$-\int_0^{\ell} \lambda_{-2}\Big[h_2(x,\nu_t(t_f,x)) - h_2(x,\nu_t^\circ(t_f,x))\Big]dx.$$

Using convexity assumption (e),

$$-\lambda_0[\mathcal{J}_0(\mathcal{C}) - \mathcal{J}_0(\mathcal{C}^\circ)] \geq$$
$$-\int_0^{\ell} \lambda_0 \left[\frac{\partial \mathcal{G}_1}{\partial \nu}(x,\nu^\circ(t_f,x)\Delta\nu(t_f,x)) + \frac{\partial \mathcal{G}_2}{\partial \nu_t}(x,\nu_t^\circ(t_f,x)\Delta\nu_t(t_f,x))\right]dx$$

$$-\int_0^{t_f}\int_0^{\ell} \sum_{i=-2}^{M} \lambda_i \frac{\partial G^i}{\partial \nu}(t,x,\nu^\circ(t,x))\Delta\nu(t,x)dtdx$$
$$+\int_0^{t_f}\int_0^{\ell} \sum_{i=-2}^{M} \lambda_i[H^i(t,x,\mathcal{C}^\circ(t,x)) - H^i(t,x,\mathcal{C}(t,x))]dtdx$$

$$-\int_0^{\ell} \lambda_{-1}\frac{\partial h_1}{\partial \nu}(x,\nu^\circ(t_f,x))\Delta\nu(t_f,x)dx - \int_0^{\ell} \lambda_{-2}\frac{\partial h_2}{\partial \nu_t}(x,\nu_t^\circ(t_f,x))\Delta\nu_t(t_f,x)dx$$

$$= -\int_0^{\ell} \lambda_0\left[\frac{\partial \mathcal{G}_1}{\partial \nu}(x,\nu^\circ(t_f,x))\Delta\nu(t_f,x) + \frac{\partial \mathcal{G}_2}{\partial \nu_t}(x,\nu_t^\circ(t_f,x))\Delta\nu_t(t_f,x)dx\right]dx$$

$$-\int_0^{t_f}\int_0^{\ell} (\mathcal{M}v)\Delta\nu(t,x)dxdt + \int_0^{t_f}\int_0^{\ell} \sum_{i=-2}^{M} \lambda_i[H^i(t,x,f^\circ(t,x)) - H^i(t,x,f(t,x))]dtdx$$

$$-\int_0^{\ell}\left[\lambda_{-1}\frac{\partial h_1}{\partial \nu}(x,\nu^\circ(t_f,x))\Delta\nu(t_f,x) + \lambda_{-2}\frac{\partial h_2}{\partial \nu_t}(x,\nu_t^\circ(t_f,x))\Delta\nu_t(t_f,x)\right]dx$$

And finally employing the **Lemma 2** and Eqs.(19)-(20), we obtain

$$-\lambda_0[\mathcal{J}_0(\mathcal{C}) - \mathcal{J}_0(\mathcal{C}^\circ)] \geq \int_0^{t_f}\int_0^{\ell}\Big\{v(t,x)[\mathcal{C}^\circ(t,x) - \mathcal{C}(t,x)] + \sum_{i=-2}^{M} \lambda_i [H^i(t,x,\mathcal{C}^\circ(t,x)) - H^i(t,x,\mathcal{C}(t,x))]\Big\} dxdt \tag{33}$$

Take into consideration that above inequality given by Eq.(33) is nonnegative because of requirement (b). Hence, following inequality is obtained

$$\mathcal{J}_0(\mathcal{C}(t,x)) - \mathcal{J}_0(\mathcal{C}^\circ(t,x)) \geq 0 \tag{34}$$

□

Hence, the proof of **Theorem 2** is completed and it is concluded that for a global minimum of the performance index functional Eq.(11), the maximum principle is also a sufficient condition.

4 Numerical Example and Discussion

Let us consider a homogeneous beam system including central host layer and two patches, whom edges are parallel to edge of the beam, perfectly bounded on the both side of the beam. At the beginning of the control duration, the beam is undeformed and at rest. The beam is exposed to external excitation force, which alerts the vibrations in the beam. By following obtained theoretical results in previous sections, the goal of this example is to find the minimum level of voltage to be applied to the piezoelectric patch actuators for suppressing undesirable vibrations in the beam optimally.

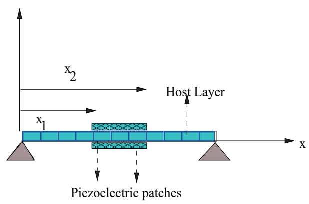

Figure 1: Cross section of the beam

The mathematical model of the beam system, described by Fig.(1), is given as follows;

$$\frac{\partial^2 \nu}{\partial x^2} + \frac{\ell^2}{12}\frac{\partial^4 \nu}{\partial x^4} + \frac{\ell^4}{36}\frac{\partial^6 \nu}{\partial x^6} = \frac{1}{c^2}\left(\frac{\partial^2 \nu}{\partial t^2} - \frac{\ell^2}{3}\frac{\rho'}{\rho}\frac{\partial^4 \nu}{\partial t^2 \partial x^2}\right) + f(t,x) + \mathcal{C}(t,x) \tag{35}$$

where $\mathcal{C}(t,x) = C(t)[H''(x-x_1) - H''(x-x_2)]$ in which $C(t)$ is the optimal control voltage function, H is heavy-side function and x_1 and x_2 are the locations of piezoelectric patch actuators. The Eq.(35) is subjected to following homogeneous boundary conditions;

$$\nu(t,0) = 0, \quad \frac{\partial^2 \nu}{\partial x^2}(t,0) = 0, \quad \frac{\partial^4 \nu}{\partial x^4}(t,0) = 0, \quad \nu(t,\ell) = 0, \quad \frac{\partial^2 \nu}{\partial x^2}(t,\ell) = 0, \quad \frac{\partial^4 \nu}{\partial x^4}(t,\ell) = 0, \tag{36}$$

or

$$\frac{\partial \nu}{\partial x}(t,0) = 0, \quad \frac{\partial^3 \nu}{\partial x^3}(t,0) = 0, \quad \frac{\partial^5 \nu}{\partial x^5}(t,0) = 0, \frac{\partial \nu}{\partial x}(t,\ell) = 0, \quad \frac{\partial^3 \nu}{\partial x^3}(t,\ell) = 0, \quad \frac{\partial^5 \nu}{\partial x^5}(t,\ell) = 0 \tag{37}$$

and initial conditions;

$$\nu(0,x) = \nu_0(x), \quad \frac{\partial \nu}{\partial t}(0,x) = \nu_1(x). \tag{38}$$

Also, the performance index functional, to be minimized at predetermined terminal time, of the system is defined as follows;

$$\mathcal{J}(C) = \int_0^\ell [\lambda_1 \nu^2(t_f, x) + \lambda_2 \nu_t^2(t_f, x)]dx + \lambda_3 \int_0^{t_f} C^2(t)dt \tag{39}$$

in which the first integral at the left hand-side is the modified kinetic energy of the beam system which includes the weighted quadratical functional of the displacement and velocity of a point on the beam. Second integral is the weighted quadratical functional of the voltage energy to be applied to the piezoelectric patch actuator on the beam system. By following the theoretical results in the previous section, the aim of this section is to optimally determine the $C(t)$, which satisfies the Eqs.(35)-(38) and minimizes the Eq.(39) in the control duration. In order to obtain the optimality for the beam system defined by Eqs.(35)-(38), let us define the adjoint system as follows;

$$\frac{\partial^2 v}{\partial x^2} + \frac{\ell^2}{12}\frac{\partial^4 v}{\partial x^4} + \frac{\ell^4}{36}\frac{\partial^6 v}{\partial x^6} = \frac{1}{c^2}\left(\frac{\partial^2 v}{\partial t^2} - \frac{\ell^2}{3}\frac{\rho'}{\rho}\frac{\partial^4 v}{\partial t^2 \partial x^2}\right). \tag{40}$$

Eq.(40) is subject to the following homogeneous boundary conditions;

$$v(t,0) = 0, \quad \frac{\partial^2 v}{\partial x^2}(t,0) = 0, \quad \frac{\partial^4 v}{\partial x^4}(t,0) = 0, \quad v(t,\ell) = 0, \quad \frac{\partial^2 v}{\partial x^2}(t,\ell) = 0, \quad \frac{\partial^4 v}{\partial x^4}(t,\ell) = 0, \tag{41a}$$

or

$$\frac{\partial v}{\partial x}(t,0) = 0, \quad \frac{\partial^3 v}{\partial x^3}(t,0) = 0, \quad \frac{\partial^5 v}{\partial x^5}(t,0) = 0, \quad \frac{\partial v}{\partial x}(t,\ell) = 0, \quad \frac{\partial^3 v}{\partial x^3}(t,\ell) = 0, \quad \frac{\partial^5 v}{\partial x^5}(t,\ell) = 0, \tag{41b}$$

and following terminal conditions;

$$\frac{1}{c^2}[\frac{\ell^2}{3}\frac{\rho'}{\rho} v_{txx}(t,x) - v_t(t,x)] = 2\lambda_1 \nu(t,x) \quad \text{at} \quad t = t_f \tag{42a}$$

$$v(t,x) = -2\lambda_2 \nu_t(t,x) \qquad \text{at} \quad t = t_f. \tag{42b}$$

In the light of obtained theoretical results from previously sections, necessary and sufficient optimality conditions in the form of maximum principle is obtained as follows;

$$\text{If} \quad \mathcal{H}[t, x_1, x_2; v^\circ, C^\circ] = \max_{C \in \mathfrak{C}_{ad}} \mathcal{H}[t, x_1, x_2; v, C] \tag{43}$$

in which the Hamiltonian is defined by the equation

$$\mathcal{H}[t, x_1, x_2; v, C] = [v_x(t, x_2) - v_x(t, x_1]C(t) - \lambda_3 C^2(t) \tag{44}$$

then,

$$\mathcal{J}[C^\circ] = \min_{C \in \mathfrak{C}_{ad}} \mathcal{J}[C], \quad C \in \mathfrak{C}_{ad}. \tag{45}$$

Hence, optimal control voltage function is obtained as follows;

$$C(t) = \frac{v_x(t, x_2) - v_x(t, x_1)}{2\lambda_3}. \tag{46}$$

The solution of the system defined by Eqs.(35)-(46) is achieved by means of `MATLAB`. Before discussing the numerical results, note that λ_3 is on the control function given by Eq.(46). The value of λ_3 on the control function increases, the value of control force defined by $C(t)$ decreases. By adjusting the optimal value of λ_3, optimal control voltage $C(t)$ is determined. In the numerical computations, $\lambda_1 = \lambda_2 = 1$, and λ_3 is evaluated as 10^{-3} and 10^3 for controlled and uncontrolled situations, respectively. Also, initial conditions are taken into account as $\nu_0(x) = \sqrt{2}\sin(\pi x)$ and $\nu_1(x) = \sqrt{2}\sin(\pi x)$. Also, the external excitation function $f(t,x) = e^{-t}(1-x)$. The predetermined terminal time is fixed as $t_f = 5$ and length of the beam is $\ell = 1$. The location of the patch on the beam is considered as $x_1 = 0.4$ and $x_2 = 0.6$. The values of the displacements and velocity in the performance index functional defined by Eq.(39), is calculated at the $x = 0.5$, which is the middle point of the beam. The Young's modulus E is 2×10^7 and the line density of the beam ρ is 6×10^4. Controlled and uncontrolled displacements of

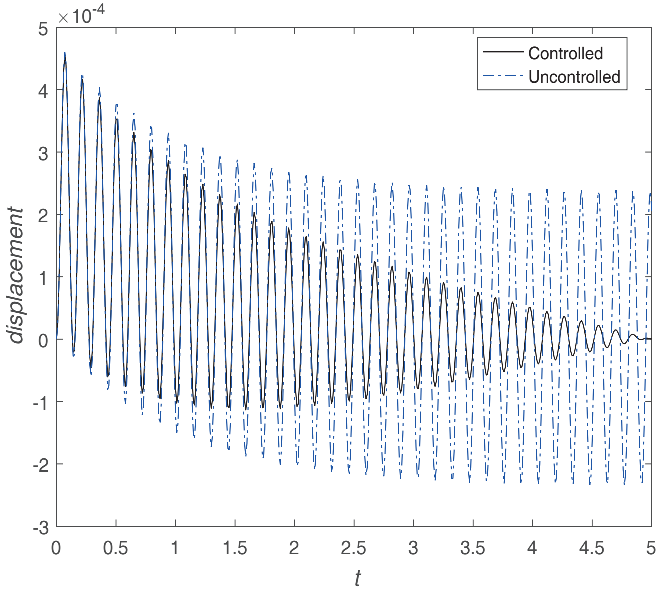

Figure 2: Controlled and uncontrolled displacements

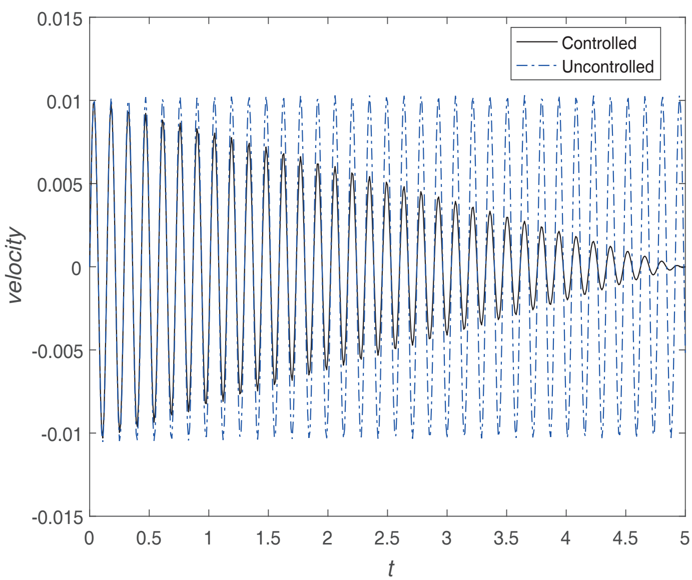

Figure 3: Controlled and uncontrolled velocities

the beam, subjected to external excitation, are plotted in Fig.2. By observing the Fig.(2), it is concluded that the vibrations in the beam is effectively suppressed as a conclusion of the optimal vibration control. Same observation is also valid for the controlled and uncontrolled velocities of the vibrations on the beam, which are plotted in Fig.(3).

Let us define the dynamic response functional of the beam as $\mathcal{J}(\nu)$ by considering $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$ in Eq.(39). Also, define the accumulated control voltage functional as $\mathcal{J}(C)$ by evaluating the $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$ Eq.(39). By taking into account the Table1, it is concluded that the value of λ_3 decreases on the optimal

Table 1: The values of $\mathcal{J}(\nu)$ and $\mathcal{J}(\mathcal{C})$ for different values of λ_3.

ϑ_3	$\mathcal{J}(w)$	$\mathcal{J}(\mathcal{V})$
10^3	3 e-5	5.2 e-10
10^0	9 e-8	2 e-6
10^{-3}	1.5 e-13	6 e-6

control function, dynamic response of the beam is decreases due to increment on the value of the optimal control force applied to piezoelectric patch actuator on beam. These observation reveal that introduced necessary and sufficient conditions in the form of maximum principle are ideal for optimality.

5 Conclusion

In this paper, optimality conditions of a hyperbolic beam equation based on Mindlin's gradient elasticity theory is studied. The system under consideration is subjected to external excitation function and nonhomogeneous boundary conditions. Also, the system has some equalities/inequalities constraints on control function and state variable. For obtaining optimality conditions of the system, existence and uniqueness of the solution to beam equation is proved by using energy-integral method and controllability of the system is discussed. Necessary and sufficient optimality conditions are derived in the form of a maximum principle. A numerical example is presented and results given by table and graphics indicate that derived conditions for a beam model based on Mindlin's gradient elasticity theory are necessary and sufficient for optimality.

6 Author Contribution

KY completed this study and wrote the manuscript. KY read and approved the final manuscript.

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8 Conflicts of Interest

This work does not have any conflicts of interest

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