Chapter 4

Notes on Tachibana Operators in the Semi-Tangent Bundle Associated with Almost R-Contact Structure

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Abstract

Studying the horizontal lifts utilizing Tachibana operators along a generalized almost r-contact structure in a semi-tangent bundle is the goal of this work. Additionally, we demonstrate several theorems on Tachibana operators with lifts and Lie derivatives.

1. Introduction

The tensor structures on smooth manifolds are remarkable geometric objects in popular differential geometry. Many authors have made important contributions to this field. In 1947, Weil noticed that there exist in a complex space a (1,1)– tensor field (i.e., an affinor field) P whose square is minus unity [24]. Ehresmann and Libermann [8] researched and provided the prerequisites for a complex structure to generate an almost complex structure. A.G. Walker began researching so-called almost product spaces in 1955 and proved that a mixed tensor field exists whose square is unity instead of being minus unity as it is in the case of an almost complex space [23]. In 1965, K. Yano tries to make as clear as possible the analogy between the almost complex and almost product structures in [26]. All the tensor

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https://doi.org/10.58830/ozgur.pub568.c2318

structures are actually a polynomial structure (P–structures). In reality, all tensor structures are polynomial structures (P–structures). Affinor fields ((1,1)– tensor fields), which are roots of the algebraic equation

$$\varphi(P) = a_1P^1 + a_2P^2 + ... + a_{(n-1)}P^{(n-1)} + a_nP^n = 0$$

in which $I \in \Im_1^1(M_n)$ is the identity tensor, can be thought of as polynomial structures $(a_1, a_2, ... a_n \in R)$. Polynomial structures on a manifold we have talked about were defined as the following equations, (*i*) If $\varphi(P) = P^2 + I = 0$, then P is referred to as an almost complex structure. As a result, we have a smooth affinor field P such that $P^2 = -I$ when regarded as a vector bundle isomorphism $P : T(M_n) \to T(M_n)$ on the tangent bundle $T(M_n)$. Thus, we defined an almost-complex structure to be a linear bundle map $P : T(M_n) \to T(M_n)$ with $P^2 = -I$. (*ii*) If $\varphi(P) = P^2 - I = 0$, then P is referred to as an almost product structure. That is to say, an almost-product structure on M_n is a field of endomorphisms of $T(M_n)$, i.e. an affinor field on M_n, so $P^2 = I$. (*iii*) If $\varphi(P) = P^2 = 0$, then P is referred to as an almost tangent structure [9]. Suppose B_m and M_n are two differentiable manifolds with dimensions m and n, respectively, and let π_1 be the submersion (natural projection) $\pi_1 : M_n \to B_m$. We may consider $(x_i) = (x^a, x^\alpha)$, $i = 1,...,n$; $a,b,... = 1,...,n-m$; $\alpha,\beta,... = n-m+1,...,n$ as local coordinates in a neighborhood $\pi_1^{-1}(U)$. Let B_m be the base manifold and $\tilde{\pi} : T(B_m) \to B_m$ be the natural projection, and let $T(B_m)$ be the tangent bundle [27] over B_m. In this case, let $T_p(B_m)$ represent in for the tangent space at a p–point ($\tilde{p} = (x^a, x^\alpha) \in M_n, p = \pi_1(\tilde{p})$) on the base manifold B_m. If $X^\alpha = dx^\alpha(X)$ are components of X in tangent space $T_p(B_m)$ with regard to the natural base $\{\partial_\alpha\} = \left\{\frac{\partial}{\partial x^\alpha}\right\}$, then we have the set of all points $(x^a, x^\alpha, x^{\bar{\alpha}})$, $X^\alpha = x^{\bar{\alpha}} = y^\alpha$, $\bar{\alpha}, \bar{\beta}, ... = n+1,...,n+m$ is by definition, the semi-tangent bundle $t(B_m)$ over the M_n manifold and the natural projection $\pi_2 : t(B_m) \to M_n$, $\dim t(B_m) = n+m$.

Specifically, assuming $n = m$, the semi-tangent bundle [17] $t(B_m)$ becomes a tangent bundle $T(B_m)$. Given a tangent bundle $\tilde{\pi} : T(B_m) \to B_m$ and a natural projection $\pi_1 : M_n \to B_m$, the pullback bundle or Whitney product (for example, see [10], [11], [17], [19], [21], [22], [29]) is given by $\pi_2 : t(B_m) \to M_n$ where

$$\begin{cases} t(B_m) = \left\{ \left(\left(x^a, x^\alpha \right), x^{\bar{\alpha}} \right) \in M_n \times T_x(B_m) : \pi_1 \left(x^a, x^\alpha \right) = \tilde{\pi} \left(x^\alpha, x^{\bar{\alpha}} \right) = \left(x^\alpha \right) \right\}, \\ t(B_m) \subset M_n \times T_x(B_m). \end{cases}$$

The induced coordinates $(x^{1'}, ..., x^{n-m'}, x^{1'}, ..., x^{m'})$ with regard to $\pi^{-1}(U)$ will be given by

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right). \end{cases} \tag{1.1}$$

If $(x^{i'}) = (x^{a'}, x^{\alpha'})$ is another coordinate chart on manifold M_n. The Jacobian matrices of (1.1) is given by [21]:

$$\left(A_j^{i'} \right) = \left(\frac{\partial x^{i'}}{\partial x^j} \right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} \\ 0 & A_\beta^{\alpha'} \end{pmatrix}$$

where $i, j = 1, ..., n.$

If the equations (1.1) is the local coordinate system of on manifold M_n, then we have the induced fiber coordinates $\left(x^a, x^\alpha, x^{\bar{\alpha}}\right)$ on the semi-tangent bundle:

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), \\ x^{\bar{\alpha}'} = \dfrac{\partial x^{\alpha'}}{\partial x^\beta} y^\beta. \end{cases} \tag{1.2}$$

The Jacobian matrices for (1.2) are as follows [21]:

$$\bar{A} = \left(A_J^{I'} \right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & A_{\beta\varepsilon}^{\alpha'} y^\varepsilon & A_\beta^{\alpha'} \end{pmatrix} \tag{1.3}$$

where $I, J = 1, ..., n + m.$

Therefore, we obtain the following matrix:

$$\left(A_{J'}^{I} \right) = \begin{pmatrix} A_{b'}^{a} & A_{\beta'}^{a} & 0 \\ 0 & A_{\beta'}^{\alpha} & 0 \\ 0 & A_{\beta'\varepsilon'}^{\alpha} y^{\varepsilon'} & A_{\beta'}^{\alpha} \end{pmatrix} \tag{1.4}$$

which is the Jacobian matrix of inverse (1.2).

We note that almost paracontact structure and almost contact structure in tangent bundles and some of their geometrical properties have been discussed in [2], [5], [7], [13], [15], [16], [18]. Several writers, including [17], [21], [22], [29] and others, have studied the differential geometry of semi-tangent bundles. It is well known that projectable linear connections in the semi-tangent bundles and their some geometrical properties were studied in [21], [22]. Several authors cited here in obtained important results in this direction [14]. Studying the horizontal lifts with Tachibana operators along a generalized almost r-contact structure (for example, see [3], [4]) in a semi-tangent bundle is the goal of this work. For Tachibana operators with lifts and Lie derivatives, we also prove several of theorems.

2. Preliminaries

By prescribing a smooth function f on base manifold B_m, we write ${}^{vv}f$ for the function f on the semi-tangent bundle $t(B_m)$ acquired by forming the composition of ${}^{v}f = f \circ \pi_1$ and $\pi : t(B_m) \to B_m$, so that

$${}^{vv}f = {}^{v}f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi.$$

Then we have

$${}^{vv}f(x^a, x^\alpha, x^{\bar{\alpha}}) = f(x^\alpha)\,. \tag{2.1}$$

The function ${}^{vv}f$ constructed is called the vertical lift of the function f to the $t(B_m)$. Here, we notice that the value of function ${}^{vv}f$ is constant for every fiber in $\pi : t(B_m) \to B_m$ [17].

The complete lift of a function $f \in M_n$ is defined as follows. Let $f = f(x^a, x^\alpha)$ be the function of this form on M_n. Then the complete lift ${}^{cc}f$ of this form is a form on $t(B_m)$ such that [17]:

$${}^{cc}f = \imath(df) = x^{\bar{\beta}}\partial_\beta f = y^\beta \partial_\beta f\,. \tag{2.2}$$

Let $X = X^\alpha \dfrac{\partial}{\partial x^\alpha}$ be local expressions in $U \subset B_m$ of a vector field $X \in \Im_0^1(B_m)$. Then the vertical lift ${}^{vv}X \in \Im_0^1(t(B_m))$ of X is given by the formula [21]:

$${}^{vv}X = X^\alpha \frac{\partial}{\partial x^{\bar{\alpha}}}, \tag{2.3}$$

relative to the natural frame

$$\left\{\frac{\partial}{\partial x^{a}},\frac{\partial}{\partial x^{\alpha}},\frac{\partial}{\partial x^{\bar{\alpha}}}\right\}.$$

By using relations (1.3) and (2.3), we can readily determine that

$${}^{vv}X' = \bar{A}({}^{vv}X).$$

Let $\omega = \omega_{\alpha} dx^{\alpha}$ be the expression of the coordinates in U subset B_m of a covector (1–form) fields $\omega \in \Im_1^0(B_m)$. On putting

$${}^{vv}\omega = \left({}^{vv}\omega\right)_{\alpha} = \left(0, \omega_{\alpha}, 0\right), \tag{2.4}$$

we can easily verify that by (1.3):

$${}^{vv}\omega = \bar{A}\,{}^{vv}\omega'.$$

The vertical lift of the covector field ω to $t(B_m)$ is the name of the $(0,1)-$tensor field ${}^{vv}\omega$ [21].

The complete lift ${}^{cc}\omega \in \Im_1^0(t(B_m))$ of $\omega \in \Im_1^0(B_m)$ with the components ω_{α} in B_m has the following components

$${}^{cc}\omega : \left(0, y^{\varepsilon}\partial_{\varepsilon}\omega_{\alpha}, \omega_{\alpha}\right) \tag{2.5}$$

relative to the induced coordinates in the semi-tangent bundle [21].

We assume that vector field $\tilde{X}$ is a projectable $(1,0)-$tensor field [22] on manifold M_n with projection $X = X^{\alpha}(x^{\alpha})\partial_{\alpha}$, i.e. $\widetilde{X} = \widetilde{X}^{a}(x^{a}, x^{\alpha})\partial_{a} + X^{\alpha}(x^{\alpha})\partial_{\alpha}$. Now, take into account $\widetilde{X} \in \Im_0^1(M_n)$, in that case complete lift ${}^{cc}\widetilde{X}$ has components of the form [17]:

$${}^{cc}\widetilde{X} = \left({}^{cc}\widetilde{X}^{\alpha}\right) = \begin{pmatrix} \widetilde{X}^{a} \\ X^{\alpha} \\ y^{\varepsilon}\partial_{\varepsilon}X^{\alpha} \end{pmatrix} \tag{2.6}$$

relative to the coordinates $(x^{a}, x^{\alpha}, x^{\bar{\alpha}})$ on the semi-tangent bundle $t(B_m)$.

For an arbitrary affinor field $F \in \Im_1^1(B_m)$, we have $(\gamma F)' = \bar{A}(\gamma F)$ in $t(B_m)$, where the matrix [14]:

$$\gamma F = (\gamma F^{I}) = \begin{pmatrix} 0 \\ 0 \\ y^{\varepsilon}F_{\varepsilon}^{\alpha} \end{pmatrix} \tag{2.7}$$

defines a (1,0)– tensor field relative to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$, and where the matrix $\bar{A}$ given with (1.3).

For each projectable $(1,0)-$tensor field $\widetilde{X} \in \Im_0^1(M_n)$ [22], we well-know that the ${}^{HH}\tilde{X}-$horizontal lift of $\tilde{X}$ to $t(B_m)$ (see [14]) by

$${}^{HH}\tilde{X} = {}^{cc}\tilde{X} - \gamma(\nabla\tilde{X}).$$

In the above situation, a differentiable manifold B_m has a projectable symmetric linear connection denoted by ∇. We recall that $\gamma(\nabla\tilde{X})-$ vector field has components [14]:

$$\gamma(\nabla\widetilde{X}) = \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon \nabla_\varepsilon X^\alpha \end{pmatrix}$$

relative to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$ on semi-tangent bundle. $\nabla_\alpha X^\varepsilon$ being the covariant derivative of X^ε, i.e.

$$(\nabla_\alpha X^\varepsilon) = \partial_\alpha X^\varepsilon + X^\beta \Gamma_{\beta\ \alpha}^{\ \varepsilon}.$$

When we contrast complete lift with horizontal lift, we get

$${}^{HH}\tilde{X} = ({}^{cc}\hat{\nabla}_X)$$

for any arbitrary projectable vector $\widetilde{X} \in \Im_0^1(M_n)$ [22], where $\hat{\nabla}$ is an affine connection in manifold B_m given by

$$\hat{\nabla}_X Y = \nabla_Y X + [X, Y]$$

or

$${}^{vv}(\hat{\nabla}_Y X) = {}^{vv}(\hat{\nabla}_X Y) + {}^{vv}[Y, X].$$

Consequently, the ${}^{HH}\tilde{X}-$horizontal lift of $\tilde{X}$ to $t(B_m)$ contains the following components [14]:

$${}^{HH}\widetilde{X} = \left({}^{HH}\widetilde{X}^I\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ -\Gamma^\alpha{}_\beta X^\beta \end{pmatrix} \quad (2.8)$$

relative to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$ on semi-tangent bundle. Where

$$\Gamma^\alpha{}_\beta = y^\varepsilon \Gamma_{\varepsilon\ \beta}^{\ \alpha}. \quad (2.9)$$

Vertical lifts are given by the following relations:

$$^{vv}\left(P\otimes Q\right)={}^{vv}P\otimes{}^{vv}Q,\quad {}^{vv}\left(P+R\right)={}^{vv}P+{}^{vv}R \tag{2.10}$$

to an algebraic isomorphism (unique) of $\Im(B_m)-$ tensor algebra into the $\Im(t(B_m))-$tensor algebra with regard to constant coefficients. Where P, Q and R being arbitrary elements of $t(B_m)$.

For an arbitrary afinor field $F\in\Im_1^1(B_m)$, if (1.3) is taken into consideration, we may demonstrate that ${}^{vv}F_J^I=A_{I'}^I A_J^{J'}({}^{vv}F_{J'}^{I'})$, where ${}^{vv}F$ is a (1,1)– tensor field defined by [21]:

$$^{vv}F=\left({}^{vv}F_J^I\right)=\begin{pmatrix}0&0&0\\0&0&0\\0&F_\beta^\alpha&0\end{pmatrix}, \tag{2.11}$$

relative to the coordinates $(x^a,x^\alpha,x^{\bar{\alpha}})$. The (1,1)– tensor field (2.11) is called the vertical lift of aff inor field F to $t(B_m)$.

Complete lifts are given by the following relations:

$$^{cc}\left(P\otimes Q\right)={}^{cc}P\otimes{}^{vv}Q+{}^{vv}P\otimes{}^{cc}Q,\quad {}^{cc}\left(P+R\right)={}^{cc}P+{}^{cc}R \tag{2.12}$$

to an algebraic isomorphism (unique) of $\Im(B_m)-$tensor algebra into the $\Im(t(B_m))-$tensor algebra with regard to constant coefficients. Where P,Q,R being arbitrary elements of $t(B_m)$.

For an arbitrary projectable afinor field $\tilde{F}\in\Im_1^1(M_n)$ [22] with projection $F=F_\beta^\alpha\left(x^\alpha\right)\partial_\alpha\otimes dx^\beta$ i.e. $\tilde{F}$ has components

$$\widetilde{F}=\left(\widetilde{F}_j^i\right)=\begin{pmatrix}\widetilde{F}_b^a(x^a,x^\alpha)&\widetilde{F}_\beta^a(x^a,x^\alpha)\\0&\widetilde{F}_\beta^\alpha(x^\alpha)\end{pmatrix}$$

relative to the coordinates (x^a,x^α). If (1.3) is taken into consideration, we may demonstrate that ${}^{cc}F_{J'}^{I'}=A_{I'}^I A_J^{J'}({}^{cc}F_{J'}^{I'})$ where ${}^{cc}\widetilde{F}$ is an affinor field (1,1)– tensor field) given by

$$^{cc}\widetilde{F}=({}^{cc}\widetilde{F}_J^I)=\begin{pmatrix}\widetilde{F}_b^a&\widetilde{F}_\beta^a&0\\0&F_\beta^\alpha&0\\0&y^\varepsilon\partial_\varepsilon F_\beta^\alpha&F_\beta^\alpha\end{pmatrix}, \tag{2.13}$$

relative to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$. The (1,1)– tensor field (2.13) is called the complete lift of afinor field to semi-tangent bundle $t(B_m)$ [21]. We will now give below some important equations that we will use:

Theorem 1. Let $X, Y \in \Im_0^1(B_m)$. If $f \in \Im_0^0(B_m)$, $\omega \in \Im_1^0(B_m)$, $F \in \Im_1^1(B_m)$ and $I = id_{B_m}$, then [21], [22]:

(i) ${}^{vv}(fX) = {}^{vv}f\,{}^{vv}X$,

(ii) ${}^{vv}I\,{}^{vv}X = 0$,

(iii) ${}^{vv}(f\omega) = {}^{vv}f\,{}^{vv}\omega$,

(iv) $\left[{}^{vv}X, {}^{vv}Y\right] = 0$,

(v) ${}^{vv}F\,{}^{vv}X = 0$,

(vi) ${}^{vv}X\,{}^{vv}f = 0$,

(vii) ${}^{vv}\omega\,{}^{vv}X = 0$.

Theorem 2. Let $\widetilde{X}, \widetilde{Y}$ and $\widetilde{F}$ be projectable vector and affinor fields on M_n with projections X, Y and F on B_m, respectively. If $f \in \Im_0^0(B_m)$, $\omega \in \Im_1^0(B_m)$ and $I = id_{B_m}$, then [21], [22]:

(i) ${}^{cc}\left(\widetilde{fX}\right) = {}^{cc}f\,{}^{vv}X + {}^{vv}f\,{}^{cc}\widetilde{X}$,

(ii) ${}^{cc}\widetilde{X}\,{}^{vv}f = {}^{vv}(Xf)$,

(iii) ${}^{vv}\omega\left({}^{cc}\widetilde{X}\right) = {}^{vv}(\omega(X))$,

(iv) ${}^{vv}X\,{}^{cc}f = {}^{vv}(Xf)$,

(v) ${}^{vv}F\,{}^{cc}\widetilde{X} = {}^{vv}(FX)$,

(vi) ${}^{vv}\omega\left({}^{cc}\widetilde{X}\right) = {}^{cc}(\omega(X))$,

(vii) ${}^{cc}\omega\left({}^{vv}X\right) = {}^{vv}(\omega(X))$,

$(viii)$ $\left[{}^{vv}X, {}^{cc}\widetilde{Y}\right] = {}^{vv}[X, Y]$,

(ix) ${}^{cc}\tilde{I} = \tilde{I}$,

(x) ${}^{vv}I\,{}^{cc}\widetilde{X} = {}^{vv}X$,

(xi) $\left[{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}\right] = {}^{cc}\left[\widetilde{X, Y}\right]$,

$$(xii)\ \ {}^{cc}\omega\left({}^{cc}\widetilde{X}\right) = {}^{cc}\left(\omega X\right),$$

$$(xiii)\ \ {}^{cc}\widetilde{X}\,{}^{cc}f = {}^{cc}\left(Xf\right),$$

$$(xiv)\ \ {}^{cc}\widetilde{F}\,{}^{vv}X = {}^{vv}\left(FX\right),$$

$$(xv)\ \ {}^{cc}\left(\widetilde{FX}\right) = {}^{cc}\widetilde{F}\,{}^{cc}\widetilde{X}.$$

Definition 1. Let $X \in \Im_0^1(M_n)$ and $T \in \Im_q^p(M_n)$. Then the classical definition of the Lie derivative of the tensor field T with respect to the vector field X is the tensor field $L_X T$ of type (p,q) with components

$$\begin{aligned}
(L_X T)^{i_1\ldots i_p}_{j_1\ldots j_q} &= X^l \partial_l T^{i_1\ldots i_p}_{j_1\ldots j_q} - T^{s_{i_2}\ldots i_p}_{j_1\ldots j_q}\partial_s X^{i_1} \\
&- T^{i_1 s_{i_3}\ldots i_p}_{j_1\ldots j_q}\partial_s X^{i_2} - \ldots \\
&+ T^{i_1\ldots i_p}_{s j_2\ldots j_q}\partial_{j_1} X^s + T^{i_1\ldots i_p}_{j_1 s\ldots j_q}\partial_{j_2} X^s + \ldots
\end{aligned}$$

for a vector fields X,Y given in manifold M_n, the Lie bracket $[X,Y]$ of X and Y is the vector field which acts on a function $f \in C^\infty(M_n)$

$$\begin{aligned}
&L_X f = Xf, \forall f \in \Im_0^0(M_n), \\
&L_X Y = [X,Y], \forall X,Y \in \Im_0^1(M_n).
\end{aligned} \tag{2.21}$$

The Lie derivative $L_X F$ of the (1,1)– afinor field F with respect to the vector field X over B_m is the field of a differential-geometric object and is given by

$$(L_X F)Y = [X, FY] - F[X,Y]. \tag{2.22}$$

Proposition 1. Let there be given a projectable tensor field, say, $\tilde{X}$, $\tilde{Y}$ of type (1,0) in M_n with projections X and Y on B_m, respectively. If f is any real valued function on B_m and L_X is the Lie derivative in the direction of the vector X, then we easily obtain [22]:

$(i)\ L_{{}^{vv}X}({}^{vv}f) = 0,$ $(ii)\ L_{{}^{vv}X}({}^{cc}f) = {}^{vv}(L_X f),$

$(iii)\ L_{{}^{cc}\tilde{X}}({}^{vv}f) = {}^{vv}(L_X f),$ $(iv)\ L_{{}^{cc}\tilde{X}}({}^{cc}f) = {}^{cc}(L_X f),$

$(v)\ L_{{}^{vv}X}({}^{vv}Y) = 0,$ $(vi)\ L_{{}^{vv}\tilde{X}}({}^{cc}\tilde{Y}) = {}^{vv}(L_X Y),$

$(vii)\ L_{{}^{cc}\tilde{X}}({}^{vv}Y) = {}^{vv}(L_X Y),$ $(viii)\ L_{{}^{cc}\tilde{X}}({}^{cc}\tilde{Y}) = {}^{cc}(\widetilde{L_X Y}).$

Definition 2. Differential transformation of algebra $t(B_m)$, given by

$D = \nabla_X : t(B_m) \to t(B_m),\ X \in \Im_0^1(B_m),$

is called as covariant derivation with respect to vector field X, if

$\nabla_{fX+gY} t = f\nabla_X t + g\nabla_Y t,$

$\nabla_X f = Xf,$

where $\forall f, g \in \Im_0^0(B_m),\ \forall X, Y \in \Im_0^1(B_m),\ \forall t \in \Im(B_m).$

However, a map defined by

$\nabla : \Im_0^1(B_m) \times \Im_0^1(B_m) \to \Im_0^1(B_m)$

is called as affine (or linear) connection [22], [25].

The concept of projectable classical linear connection as follows. Let $p : Y \to M$ be a fibred manifold. A classical connection ∇ on Υ is said to be projectable (with respect to p) if there is a unique classical linear connection $\underline{\nabla}$ on M such that ∇ and $\underline{\nabla}$ are p–related [1], [25]. In particular, if $T(B_m)$ is the tangent bundle of base manifold B_m, then a linear connection $\underline{\nabla}$ is a classical linear connection on manifold B_m [12]. The last condition means that if $\tilde{X}, \tilde{Y} \in \Im_0^1(M_n)$ and $X, Y \in \Im_0^1(B_m)$ are such that $T_p \circ \tilde{X} = X \circ p$ and $T_p \circ \tilde{Y} = Y \circ p$ then $T_p \circ \nabla_{\tilde{X}} \tilde{Y} = (\underline{\nabla}_X Y) \circ p.$

T is determined by

$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$

for any $X, Y \in \Im_0^1(B_m)$). Along with the above concept ∇ is a projectable linear connection on B_m (with respect to $p := \pi_1 : M_n \to B_m$).

Theorem 3. Let $\widetilde{X}, \widetilde{Y}$ and $\widetilde{F}$ be projectable vector and affinor fields on M_n with projections X, Y and F on B_m, respectively. If $f \in \Im_0^0(B_m)$, $\omega \in \Im_1^0(B_m)$ and $I = id_{B_m}$, then [22]:

$(i)\ {}^{HH}\tilde{I} = I,$

$(ii)\ {}^{HH}\tilde{I}\,{}^{vv}X = {}^{vv}X,$

$(iii)\ {}^{vv}I\,{}^{HH}\widetilde{X} = {}^{vv}X,$

$(iv)\ {}^{HH}\tilde{I}\,{}^{HH}\widetilde{X} = {}^{HH}\widetilde{X},$

$(v)\ {}^{HH}\widetilde{X}\,{}^{vv}f = {}^{vv}(Xf),$

$(vi)\ {}^{HH}(fX) = {}^{vv}f\, {}^{HH}\widetilde{X}$,

$(vii)\ {}^{HH}\omega\left({}^{HH}\widetilde{X}\right) = 0$,

$(viii)\ {}^{vv}\omega\left({}^{HH}\widetilde{X}\right) = {}^{vv}\left(\omega(X)\right)$,

$(ix)\ {}^{HH}\omega\left({}^{vv}X\right) = {}^{vv}\left(\omega(X)\right)$,

$(x)\ {}^{HH}\widetilde{F}\, {}^{vv}X = {}^{vv}\left(FX\right)$,

$(xi)\ {}^{HH}\widetilde{F}\, {}^{HH}\widetilde{X} = {}^{HH}\left(\widetilde{FX}\right)$.

Definition 3. Let $\tilde{X}$ and $\tilde{Y}$ be projectable vector fields on M_n with projections X and Y on B_m, respectively. For the Lie product, we obtain [29]:

$(i)\ \left[{}^{vv}X, {}^{HH}\widetilde{Y}\right] = {}^{vv}[X,Y] - {}^{vv}\left(\nabla_X Y\right) = -{}^{vv}\left(\hat{\nabla}_Y X\right)$,

$(ii)\ \left[{}^{cc}\widetilde{X}, {}^{HH}\widetilde{Y}\right] = {}^{HH}\widetilde{[X,Y]} - \gamma\left(L_X Y\right)$,

$(iii)\ \left[{}^{HH}\widetilde{X}, {}^{vv}Y\right] = {}^{vv}[X,Y] + {}^{vv}\left(\nabla_Y X\right)$,

$(iv)\ \left[{}^{HH}\widetilde{X}, {}^{HH}\widetilde{Y}\right] = {}^{HH}\widetilde{[X,Y]} - \gamma\hat{R}\left(X,Y\right)$,

where the curvature tensor of the affine connection $\hat{\nabla}$ is represented by $\hat{R}$.

Definition 4.

Assume that ∇ in B_m is a projectable linear connection. We will use the following conditions to define the horizontal lift ${}^{HH}\nabla$ of a projectable linear connection ∇ in B_m to $t(B_m)$ [28], [29]:

$(i)\ {}^{HH}\nabla_{{}^{vv}X}\, {}^{vv}Y = 0$,

$(ii)\ {}^{HH}\nabla_{{}^{vv}X}\, {}^{HH}\widetilde{Y} = 0$,

$(iii)\ {}^{HH}\nabla_{{}^{HH}\widetilde{X}}\, {}^{vv}Y = {}^{vv}\left(\nabla_X Y\right)$,

$(iv)\ {}^{HH}\nabla_{{}^{HH}\widetilde{X}}\, {}^{HH}\widetilde{Y} = {}^{HH}\widetilde{\left(\nabla_X Y\right)}$,

for any $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$.

Proposition 2.

Let $\widetilde{S}$ and $\widetilde{T}$ be two tensor fields of type (r,s) in $t(B_m)$ such that

$$\widetilde{S}(\widetilde{X}_s,...,\widetilde{X}_1)=\widetilde{T}(\widetilde{X}_s,...,\widetilde{X}_1)$$

for all vector fields $\widetilde{X}_t$ $(t=1,2,...,s)$ which are of the form $\overline{\overline{X}}$, ${}^{vv}X$ or ${}^{HH}\widetilde{X}$, where $X\in\Im_0^1(M_n)$. Then $\widetilde{S}=\widetilde{T}$ (for example, see [27]).

3. Main Results

3.1. Tachibana Operators for Generalized Almost R-Contact Structure in Semi-tangent Bundle

Let B_m be a differentiable manifold of C^∞ class and $T(B_m)$ denotes the semi-tangent bundle of B_m $(m=2k+r,\ k\geq 0)$. Suppose that there are projectable affinor field $\tilde{\phi}\in\Im_1^1(M_n)$ [22] with projection $\phi=\phi_\beta^\alpha\left(x^\alpha\right)\partial_\alpha\otimes dx^\beta$ i.e., a projectable $(1,0)-$tensor field $\tilde{\xi}_p\in\Im_0^1(M_n)$ with projection $\xi_p=\xi^\alpha\left(x^\alpha\right)\partial_\alpha$ i.e. $\tilde{\xi}_p=\tilde{\xi}^a(x^\alpha,x^\alpha)\partial_a+\xi^\alpha\left(x^\alpha\right)\partial_\alpha$ [22], a covector field η_p, $p=1,2,....r$ satisfying (for example, see [2], [5], [20]):

$$(i)\ \ \phi^2=a^2I+\in\sum\nolimits_{p=1}^{r}\xi_p\otimes\eta_p$$

$$(ii)\ \ \phi\xi_p=0$$

$$(iii)\ \ \eta_p\circ\phi=0$$

$$(iv)\ \ \eta_p\left(\xi_q\right)=-\frac{a^2}{\in}\delta_{pq}. \tag{3.1}$$

Where a and $\in$ are non-zero complex numbers and $p=1,2,...,r$ and δ_{pq} represent the Kronecker delta. A generalized almost r-contact manifold with a generalized almost r-contact structure, or simply an $(\phi,\eta_p,\xi_p,a,\in)$-structure, is what the manifold B_m is known as.

Let B_m be the base space where the Lorentzian almost r-para-contact structure is accepted. Then there exists a projectable affinor field $\tilde{\phi}$ of type $(1,1)$, $r\left(C^\infty\right)$ vector fields $\tilde{\xi}_1,\tilde{\xi}_2,...\tilde{\xi}_p$, and $r\left(C^\infty\right)$ 1-forms $\eta_1,\eta_2,...,\eta_p$, such that equation (3.1) are satisfied. We get the following by taking the complete lifts of equation (3.1):

$$(i)\ \ \left(\phi^H\right)^2=a^2I+\in\sum\nolimits_{p=1}^{r}\left\{\xi_p^V\otimes\eta_p^H+\xi_p^H\otimes\eta_p^V\right\}$$

$(ii)\ {}^{HH}\phi\, {}^{vv}\xi_p = 0,\ {}^{HH}\phi\, {}^{cc}\xi_p = 0$

$(iii)\ {}^{vv}\eta_p \circ {}^{HH}\phi = 0,\ {}^{HH}\eta_p \circ {}^{vv}\phi = 0,\ {}^{HH}\eta_p \circ {}^{HH}\phi = 0,\ {}^{vv}\eta_p \circ {}^{vv}\phi = 0$

$$(iv)\ {}^{HH}\eta_p\left({}^{HH}\xi_p\right) = {}^{vv}\eta_p\left({}^{vv}\xi_p\right) = 0,\ {}^{HH}\eta_p\left({}^{vv}\xi_p\right) = {}^{vv}\eta_p\left({}^{HH}\xi_p\right) = -\frac{a^2}{\in}\delta_{pq} \quad (3.2)$$

Let's use

$$\tilde{J} = \phi^H + \frac{\in}{a}\sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right) \quad (3.3)$$

to define the $\tilde{J}$ element of $J_0^1 t\left(B_m\right)$.

Then, considering equation (3.2), it is evident that

$\tilde{J}^2\, {}^{vv}X = a^2\, {}^{vv}X,\ \tilde{J}^2\, {}^{HH}\widetilde{X} = a^2\, {}^{HH}\widetilde{X}$

which gives that $\tilde{J}$ is GF structure in $t\left(B_m\right)$ (for example, see [6], [7]).

Considering Equation (3.4), we now have

$$(i)\ \tilde{J}\, {}^{HH}\widetilde{X} = {}^{HH}\left(\widetilde{\phi X}\right) + \frac{\in}{a}\sum_{p=1}^{r}\left\{{}^{vv}\left(\eta_P\left(X\right)\right){}^{vv}\xi_p\right\}$$

$$(ii)\ \tilde{J}\, {}^{vv}X = {}^{vv}\left(\phi X\right) + \frac{\in}{a}\sum_{p=1}^{r}\left\{{}^{vv}\left(\eta_p\left(X\right)\right){}^{HH}\widetilde{\xi_p}\right\} \quad (3.4)$$

for all $X \in \Im_0^1\left(B_m\right)$.

3.2. Tachibana operator

Definition 5.

Let $\varphi \in \Im_1^1(B_m)$ and $\Im\left(B_m\right) = \sum_{r,s=0}^{\infty} \Im_s^r\left(B_m\right)$ be an tensor algebra over $\mathbb{R}$. A map $\phi_\varphi\Big|_{r+s\to 0} : \overset{*}{\Im}\left(B_m\right) \to \Im\left(B_m\right)$ is called a Tachibana operator or ϕ_φ operator on B_m if

a) ϕ_φ is linear with respect to constant coefficient,

b) $\phi_{\varphi} : \overset{*}{\Im}(B_m) \to \Im^r_{s+1}(B_m)$ for all *r* and *s*,

c) $\phi_{\varphi}\left(K \overset{C}{\otimes} L\right) = \left(\phi_{\varphi} K\right) \otimes L + K \otimes \phi_{\varphi} L$ for all $K, L \in \overset{*}{\Im}(B_m)$,

d) $\phi_{\varphi X} Y = -\left(L_Y \varphi\right) X$ for all $X, Y \in \Im^1_0(B_m)$, where L_Y is the Lie derivation with respect to *Υ*,

e) $\left(\phi_{\varphi X}\eta\right) Y = (d(\iota_Y \eta(\phi X) - (d(\iota_Y(\eta \circ \phi) X + \eta((L_Y \varphi) X)$

$$= (\phi X(\iota Y\eta))(\phi X) - X\left(\iota_{\varphi Y}\eta\right) + \eta((L_Y\varphi)X) \tag{3.5}$$

for all $\eta \in \Im^0_1(B_m)$ and $X, Y \in \Im^0_1(B_m)$, where $\acute{y}_Y \eta = \eta(Y) = \eta \overset{c}{\otimes} Y$, $\overset{*}{\Im}{}^r_s(B_m)$ the module of all pure tensor fields of type (r,s) on B_m with respect to the affinor field $\widetilde{\varphi}$ [12?][13?].

Theorem 3.1.

For the Tachibana operator on B_m, L_X the operator Lie derivation with respect to *X*, $\widetilde{J} \in \Im^1_1(t(B_m))$ defined by $\tilde{J} = {}^{HH}\tilde{\phi} + \frac{\in}{a}\sum_{p=1}^{r}\left({}^{vv}\xi_p \otimes {}^{vv}\eta_p + {}^{HH}\widetilde{\xi_p} \otimes {}^{HH}\eta_p\right)$ and $\eta(Y) = 0$, we have

$$(i)\ \Phi_{\tilde{J}\,{}^{vv}Y}\,{}^{HH}\widetilde{X} = -{}^{vv}\left(\left(\hat{\nabla}_X \phi\right) Y\right) - \frac{\in}{a}\sum_{p=1}^{r} {}^{vv}\left(\left(\hat{\nabla}_X \eta_p\right) Y\right)^{HH} \widetilde{\xi_p}$$

$$(ii)\ \Phi_{\tilde{J}\,{}^{HH}\tilde{Y}}\,{}^{HH}\widetilde{X} = -{}^{HH}\left(\widetilde{(L_X\phi)Y}\right) + \gamma\hat{R}(X, \phi Y) + \frac{\in}{a}\sum_{p=1}^{r} {}^{vv}\left(\left(L_X \eta_p\right) Y\right)^{vv}\xi_p - \tilde{J}\gamma\hat{R}(X, Y)$$

$$(iii)\ \Phi_{\tilde{J}\,{}^{vv}Y}\,{}^{vv}X = 0$$

$$(iv)\ \Phi_{\tilde{J}\,{}^{HH}\tilde{Y}}\,{}^{vv}X = -{}^{vv}\left(\left(L_X Y\right)\phi\right) + {}^{vv}\left(\left(\nabla_X \phi\right) Y\right)$$

$$-\frac{\in}{a}\sum_{p=1}^{r} {}^{vv}\left(\left(L_X \eta_p\right) Y\right)^{HH} \widetilde{\xi_p} + \frac{\in}{a}\sum_{p=1}^{r} {}^{vv}\left(\left(\nabla_X \eta_p\right) Y\right)^{HH} \widetilde{\xi_p}\,. \tag{3.6}$$

where projectable vector fields $\widetilde{X},\widetilde{Y},\widetilde{\xi_p}\in\Im_0^1(M_n)$, a projectable $(1,1)-$tensor field $\tilde{\phi}\in\Im_1^1(M_n)$ and a 1-form $\eta\in\Im_1^0(M_n)$.

Proof.

$$(i)\ \Phi_{\tilde{J}\,{}^{vv}Y}{}^{HH}\widetilde{X}=-\left(L_{{}^{HH}\widetilde{X}}\tilde{J}\right){}^{vv}Y=-\left(L_{{}^{HH}\widetilde{X}}\tilde{J}\,{}^{vv}Y-\tilde{J}L_{{}^{HH}\widetilde{X}}{}^{vv}Y\right)$$

$$=-\left[{}^{HH}\widetilde{X},{}^{vv}(\varphi Y)+\frac{\in}{a}\sum_{p=1}^{r}{}^{HH}\left(\widetilde{\eta_p(Y)\xi_p}\right)\right]+\left({}^{HH}\tilde{\varphi}+\frac{\in}{a}\sum_{p=1}^{r}\left({}^{vv}\xi_p\otimes{}^{vv}\eta_p+{}^{HH}\widetilde{\xi_p}\otimes{}^{HH}\eta_p\right)\right)\underbrace{\left[{}^{HH}\widetilde{X},{}^{vv}Y\right]}_{{}^{vv}[X,Y]+{}^{vv}(\nabla_Y X)}$$

$$=-\underbrace{\left[{}^{HH}\widetilde{X},{}^{vv}(\varphi Y)\right]}_{{}^{vv}[X,\varphi Y]+{}^{vv}(\nabla_{\varphi Y}X)}-\left[{}^{HH}\widetilde{X},\frac{\in}{a}\sum_{p=1}^{r}{}^{HH}\left(\widetilde{\underbrace{\eta_p(Y)}_{0}\xi_p}\right)\right]$$

$$+{}^{HH}\tilde{\varphi}\left[{}^{HH}\widetilde{X},{}^{vv}Y\right]+\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\eta_p\left(\left[{}^{HH}\widetilde{X},{}^{vv}Y\right]\right){}^{vv}\xi_p+\frac{\in}{a}\sum_{p=1}^{r}{}^{HH}\eta_p\left(\left[{}^{HH}\widetilde{X},{}^{vv}Y\right]\right){}^{HH}\widetilde{\xi_p}$$

$$=-{}^{vv}\left((L_X\varphi)Y\right)-{}^{vv}\left(\varphi(L_XY)\right)-{}^{vv}\left(\hat{\nabla}_X\varphi Y\right)-{}^{vv}[\varphi Y,X]+{}^{vv}(\varphi L_XY)$$

$$+{}^{HH}\tilde{\varphi}\,{}^{vv}(\nabla_YX)+\frac{\in}{a}\sum_{p=1}^{r}\underbrace{{}^{vv}\eta_p\,{}^{vv}[X,Y]}_{0}{}^{vv}\xi_p+\frac{\in}{a}\sum_{p=1}^{r}\underbrace{{}^{vv}\eta_p\,{}^{vv}(\nabla_YX)}_{0}{}^{vv}\xi_p$$

$$+\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left(\eta_p[X,Y]\right){}^{HH}\widetilde{\xi_p}+\frac{\in}{a}\sum_{p=1}^{r}{}^{HH}\eta_p\,{}^{vv}(\nabla_YX)\,{}^{HH}\widetilde{\xi_p}$$

$$-\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left((L_X\eta_p)Y\right){}^{HH}\widetilde{\xi_p}+\frac{\in}{a}\sum_{p=1}^{r}\left({}^{HH}\eta_p\left({}^{vv}\left(\hat{\nabla}_XY\right)+{}^{vv}[Y,X]\right)\right){}^{HH}\widetilde{\xi_p}$$

$$=-{}^{vv}\left(\left(\hat{\nabla}_X\varphi\right)Y\right)-{}^{vv}\left(\varphi\hat{\nabla}_XY\right)+{}^{vv}\left(\varphi\left(\hat{\nabla}_XY\right)\right)-\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left((L_X\eta_p)Y\right){}^{HH}\widetilde{\xi_p}$$

$$-\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left(\left(\hat{\nabla}_X\eta_p\right)Y\right){}^{HH}\widetilde{\xi_p}+\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left((L_X\eta_p)Y\right){}^{HH}\widetilde{\xi_p}$$

$$=-{}^{vv}\left(\left(\hat{\nabla}_X\varphi\right)Y\right)-\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left(\left(\hat{\nabla}_X\eta_p\right)Y\right){}^{HH}\widetilde{\xi_p},\tag{3.7}$$

$$(ii)\ \Phi_{\tilde{J}\,{}^{HH}\widetilde{Y}}{}^{HH}\widetilde{X}=-\left(L_{{}^{HH}\widetilde{X}}\tilde{J}\right){}^{HH}\widetilde{Y}=-\left(L_{{}^{HH}\widetilde{X}}\tilde{J}\,{}^{HH}\widetilde{Y}-\tilde{J}L_{{}^{HH}\widetilde{X}}{}^{HH}\widetilde{Y}\right)$$

$$=-\left[{}^{HH}\widetilde{X},{}^{HH}\left(\widetilde{\varphi Y}\right)+{}^{vv}\left(\eta_p(Y)\xi_p\right)\right]+\left({}^{HH}\tilde{\varphi}+\frac{\in}{a}\sum_{p=1}^{r}\left({}^{vv}\xi_p\otimes{}^{vv}\eta_p+{}^{HH}\widetilde{\xi_p}\otimes{}^{HH}\eta_p\right)\right)\left[{}^{HH}\widetilde{X},{}^{HH}\widetilde{Y}\right]$$

$$= -{}^{HH}\widetilde{\left(\left(L_X\varphi\right)Y\right)} + \gamma\hat{R}\left(X,\varphi Y\right) + \frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left(\underbrace{L_X\eta_p\left(Y\right)}_{0}\right){}^{vv}\xi_p$$

$$-\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left(\left(L_X\eta_p\right)Y\right){}^{vv}\xi_p - J\left(\gamma\hat{R}\left(X,Y\right)\right)$$

$$= -{}^{HH}\widetilde{\left(\left(L_X\varphi\right)Y\right)} + \gamma\hat{R}\left(X,\varphi Y\right) - \frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left(\left(L_X\eta_p\right)Y\right){}^{vv}\xi_p - J\left(\gamma\hat{R}\left(X,Y\right)\right), \quad (3.8)$$

$$(iii)\quad \Phi_{\tilde{J}\,{}^{vv}Y}\,{}^{vv}X = -\left(L_{{}^{vv}X}\tilde{J}\right){}^{vv}Y = -\left(L_{{}^{vv}X}\tilde{J}\,{}^{vv}Y - \tilde{J}\underbrace{L_{{}^{vv}X}\,{}^{vv}Y}_{0}\right)$$

$$= -\left(L_{{}^{vv}X}\tilde{J}\,{}^{vv}Y\right) = -\left[{}^{vv}X, \left({}^{HH}\tilde{\phi} + \frac{\in}{a}\sum_{p=1}^{r}\left({}^{vv}\xi_p \otimes {}^{vv}\eta_p + {}^{HH}\widetilde{\xi_p} \otimes {}^{HH}\eta_p\right)\right){}^{vv}Y\right]$$

$$= -\left[{}^{vv}X, \frac{\in}{a}\sum_{p=1}^{r}\left({}^{vv}\eta_p\,{}^{vv}Y\right){}^{vv}\xi_p\right] - \left[{}^{vv}X, \frac{\in}{a}\sum_{p=1}^{r}\left({}^{HH}\eta_p\,{}^{vv}Y\right){}^{HH}\widetilde{\xi_p}\right]$$

$$= -\left[{}^{vv}X, \frac{\in}{a}\sum_{p=1}^{r}\underbrace{\left({}^{vv}\eta_p\,{}^{vv}Y\right)}_{0}{}^{vv}\xi_p\right] - \left[{}^{vv}X, \frac{\in}{a}\sum_{p=1}^{r}\underbrace{\left({}^{HH}\eta_p\,{}^{vv}Y\right)}_{{}^{vv}\left(\eta_p(Y)\right)}{}^{HH}\widetilde{\xi_p}\right]$$

$$= -\left[{}^{vv}X, \frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\underbrace{\left(\eta_p\left(Y\right)\right)}_{0}{}^{HH}\widetilde{\xi_p}\right] = 0 \quad (3.9)$$

$$(iv)\quad \Phi_{\tilde{J}\,{}^{HH}\widetilde{Y}}\,{}^{vv}X = -\left(L_{{}^{vv}X}\tilde{J}\right){}^{HH}\widetilde{Y} = -\left(L_{{}^{vv}X}\tilde{J}\,{}^{HH}\widetilde{Y} - \tilde{J}L_{{}^{vv}X}\,{}^{HH}\widetilde{Y}\right)$$

$$= -{}^{vv}\left[X,\phi Y\right] + {}^{vv}\left(\nabla_X\phi Y\right) + \underbrace{{}^{HH}\tilde{\phi}\,{}^{vv}\left[X,Y\right]}_{{}^{vv}\left(\phi L_X Y\right)} - \underbrace{{}^{HH}\tilde{\phi}\,{}^{vv}\left(\nabla_X Y\right)}_{{}^{vv}\left(\phi\nabla_X Y\right)} + \frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\eta_p\left({}^{vv}\left[X,Y\right] - {}^{vv}\left(\nabla_X Y\right)\right){}^{vv}\xi_p$$

$$+\frac{\in}{a}\sum_{p=1}^{r}{}^{HH}\eta_p\left({}^{vv}\left[X,Y\right] - {}^{vv}\left(\nabla_X Y\right)\right){}^{HH}\widetilde{\xi_p}$$

$$= -{}^{vv}\left(\left(L_X Y\right)\phi\right) + {}^{vv}\left(\left(\nabla_X\phi\right)Y\right) + \frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left(\underbrace{L_X\eta_p\left(Y\right)}_{0} - \left(L_X\eta_p\right)Y\right){}^{HH}\widetilde{\xi_p}$$

$$-\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left(\underbrace{\nabla_X\eta_p(Y)-(\nabla_X\eta_p)Y}_{0}\right)^{HH}\widetilde{\xi_p}$$

$$=-{}^{vv}((L_XY)\phi)+{}^{vv}((\nabla_X\phi)Y)-\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left((L_X\eta_p)Y\right)^{HH}\widetilde{\xi_p}+\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left((\nabla_X\eta_p)Y\right)^{HH}\widetilde{\xi_p} \quad (3.10)$$

where $\eta_p L_X Y = L_X\eta_p(Y)-(L_X\eta_p)Y$ and $\eta_p\nabla_X Y=\nabla_X\eta_p(Y)-(\nabla_X\eta_p)Y$.

Corollary 3.1.

If we put $Y=\xi_p$ i.e. ${}^{HH}\eta_p\left({}^{HH}\widetilde{\xi_p}\right)={}^{vv}\eta_p\left({}^{vv}\xi_p\right)=0$, ${}^{HH}\eta_p\left({}^{vv}\xi_p\right)={}^{vv}\eta_p\left({}^{HH}\widetilde{\xi_p}\right)=-\frac{a^2}{\in}$ then we have

$$(i)\ \Phi_{\tilde{J}\,{}^{vv}\xi_p}{}^{HH}\widetilde{X}=a\sum_{p=1}^{r}{}^{HH}\widetilde{\left(L_{\xi_p}X\right)}-a\gamma\hat{R}(X,\xi_p)-{}^{vv}\left(\hat{\nabla}_X\phi\right)+\left(\hat{\nabla}_X\eta_p\right){}^{vv}\xi_p{}^{HH}\widetilde{\xi_p}$$

$$(ii)\ \Phi_{\tilde{J}\,{}^{HH}\widetilde{\xi_p}}{}^{HH}\widetilde{X}=a{}^{vv}\left(\hat{\nabla}_X\xi_p\right)-{}^{HH}\widetilde{\left((L_X\phi)\xi_p\right)}+{}^{HH}\tilde{\phi}\gamma\hat{R}(X,\xi_p)-\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left((L_X\eta_p)\xi_p\right){}^{vv}\xi_p$$

$$-\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\eta_p\gamma\hat{R}(X,\xi_p){}^{vv}\xi_p-\frac{\in}{a}\sum_{p=1}^{r}{}^{HH}\eta_p\gamma\hat{R}(X,\xi_p){}^{HH}\widetilde{\xi_p}$$

$$(iii)\ \Phi_{\tilde{J}\,{}^{vv}\xi_p}{}^{vv}X=-a{}^{vv}\left(\hat{\nabla}_{\xi_p}X\right)$$

$$(iv)\ \Phi_{\tilde{J}\,{}^{HH}\widetilde{\xi_p}}{}^{vv}X=-{}^{vv}((L_X\phi)\xi_p)+{}^{vv}((\nabla_X\phi)\xi_p)-\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left((L_X\eta_p)\xi_p\right)^{HH}\widetilde{\xi_p}+\frac{\in}{a}\sum_{p=1}^{r}{}^{vv}\left((\nabla_X\eta_p)\xi_p\right)^{HH}\widetilde{\xi_p}.$$

Acknowledgment. This study was supported by Scientific and Technological Research Council of Turkey (TUBITAK) under the Grant Number (TBAG-1001, MFAG-122F131). The authors thank to TUBITAK for their supports.

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