Chapter 2

On Lie Differentiability Conditions of Some Polynomial Structures in Semi Tangent Bundle

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Abstract

In popular differential geometry, the tensor structures on smooth manifolds are remarkable geometric objects. In reality, every tensor structure is a polynomial structure. A tensor field f of type $(1,1)$ on a differentiable manifold is called a polynomial structure if it satisfies the algebraic equation $f^n + a_1 f^{(n-1)} + ... + a_{n-1} f + a_n I = 0$, where I is the identity tensor of type $(1,1)$ and $a_1, a_2, ..., a_n$ are real numbers. The Lie differentiability conditions of some polynomial structures (almost contact and almost paracontact structures) in semi-tangent bundle t(M) are examined in this study.

1. Introduction

Let B_m and M_n denote two differentiable manifolds of dimensions m and n respectively, let $\left(M_n, \pi_1, B_m\right)$ be a differentiable bundle, and let π_1 be the submersion $\pi_1 : M_n \to B_m$. We may consider $(x^i) = (x^a, x^\alpha)$, $i = 1,...,n$; $a, b,... = 1,...,n-m$; $\alpha, \beta,... = n-m+1,...,n$ as local coordinates in a neighborhood $\pi_1^{-1}(U)$. Let B_m be the base manifold and $\tilde{\pi} : T(B_m) \to B_m$ be the natural projection, and let $T(B_m)$ be the tangent bundle over B_m. In this case, let $T_p(B_m)$ represent in for the tangent

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space at a $p-$point $(p=x^{a},x^{\alpha})\in M_{n}, p=\pi_{1}(\tilde{p}))$ on the base manifold B_m. If $X^{\alpha}=dx^{\alpha}(X)$ are components of X in tangent space $T_p(B_m)$ with respect to the natural base $\{\partial_{\alpha}\}=\{\frac{\partial}{\partial x^{\alpha}}\}$, then we have the set of all points, $(x^{a},x^{\alpha},x^{\bar{\alpha}})$, $X^{\alpha}=x^{\bar{\alpha}}=y^{\alpha}$, $\bar{\alpha},\bar{\beta},...=n+1,...,n+m$, is by definition, the semi-tangent bundle $t(B_m)$ over the M_n manifold and the natural projection $\pi_2:t(B_m)\to M_n, \dim t(B_m)=n+m$. Specifically, assuming $n=m$, the semi-tangent bundle [14] $t(B_m)$ becomes a tangent bundle $T(B_m)$. Given a tangent bundle $\tilde{\pi}:T(B_m)\to B_m$ and a natural projection $\pi_1:M_n\to B_m$, the pullback bundle (for example see [4], [5], [7], [8], [9], [10], [17], [19], [20] is given by $\pi_2:t(B_m)\to M_n$ where

$$t(B_m)=\left\{\left(\left(x^{a},x^{\alpha}\right),x^{\bar{\alpha}}\right)\in M_n\times T_x(B_m)\middle|\pi_1\left(x^{a},x^{\alpha}\right)=\tilde{\pi}\left(x^{\alpha},x^{\bar{\alpha}}\right)\right\}.$$

The induced coordinates $\left(x^{1'},...,x^{n-m'},x^{1'},...,x^{m'}\right)$ with regard to $\pi^{-1}(U)$ will be given by

$$\begin{cases} x^{a'}=x^{a'}(x^{b},x^{\beta}), & a,b,...=1,...,n-m \\ x^{\alpha'}=x^{\alpha'}\left(x^{\beta}\right), & \alpha,\beta,...=n-m+1,...,n. \end{cases} \quad (1)$$

If $(x^{i'})=(x^{a'},x^{\alpha'})$ is another coordinate chart on M_n, the Jacobian matrices of (1) is given by [14]:

$$\left(A_{j}^{i'}\right)=\left(\frac{\partial x^{i'}}{\partial x^{j}}\right)=\begin{pmatrix} \frac{\partial x^{a'}}{\partial x^{b}} & \frac{\partial x^{a'}}{\partial x^{\beta}} \\ 0 & \frac{\partial x^{\alpha'}}{\partial x^{\beta}} \end{pmatrix},$$

where $i,j,....=1,...,n.$

If (1) is local coordinate system on M_n, then we have the induced fibre coordinates $\left(x^{a'},x^{\alpha'},x^{\bar{\alpha}'}\right)$ on the semi-tangent bundle (change of coordinates):

$$\begin{cases} x^{a'}=x^{a'}(x^{b},x^{\beta}), & a,b,...=1,...,n-m, \\ x^{\alpha'}=x^{\alpha'}\left(x^{\beta}\right), & \alpha,\beta,...=n-m+1,...,n, \\ x^{\bar{\alpha}'}=\frac{\partial x^{\alpha'}}{\partial x^{\beta}}y^{\beta}, & \bar{\alpha},\bar{\beta},...=n+1,...,n+m, \end{cases} \quad (2)$$

The Jacobian matrices for (2) are as follows [14]:

$$\bar{A} = \left(A_J^{I'}\right) = \begin{pmatrix} \frac{\partial x^{a'}}{\partial x^b} & \frac{\partial x^{a'}}{\partial x^\beta} & 0 \\ 0 & \frac{\partial x^{\alpha'}}{\partial x^\beta} & 0 \\ 0 & y^\varepsilon \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\varepsilon} & \frac{\partial x^{\alpha'}}{\partial x^\beta} \end{pmatrix}, \qquad (3)$$

where $I, J, ... = 1, ..., n+m.$ Then, we obtain

$$\left(A_{J'}^{I}\right) = \begin{pmatrix} A_{b'}^{a} & A_{\beta'}^{a} & 0 \\ 0 & A_{\beta'}^{\alpha} & 0 \\ 0 & A_{\beta'\varepsilon'}^{\alpha} y^{\varepsilon'} & A_{\beta'}^{\alpha} \end{pmatrix}, \qquad (4)$$

which is the Jacobian matrix of inverse (2).

This work examines lifts of various geometric objects (complete etc. lifts of tensor fields), which were previously examined in tangent bundles, as well as their applications in semi-tangent bundles. The theory of tangent bundles [21], which is popular in engineering, physics, and particularly differential geometry, has been the subject of a great deal of research [15]. The semi-tangent bundle taken into account in this work differs from the tangent bundle in that it specifies a pull-back bundle. The almost paracontact structure and almost contact structure in tangent bundles, as well as some of their properties, have been examined in [1], [2], [3], [7], [12], [13] and [16]. Numerous authors have researched the geometric characteristics of the semi-tangent bundle, including [14], [19], [20] and others. In [9], [19] and [20], it is known that projectable linear connections in semi-tangent bundles and some of their features have been studied.

The definition of Lie derivation and its most significant properties for semi-tangent bundle are introduced in the second part. The analysis of Lie derivates of geometric structures in relation to the complete and vertical lift of $(1,0)-$tensor field X for semi-tangent bundle is described in the final part, which is the most crucial for the progress of the current inquiry. The additional data on Lie derivatives of the created geometric structures will be extensively used in future studies. Our aim is to examine Lie differentiability conditions of almost paracontact structure and almost contact structure in semi-tangent bundle tM.

2. Preliminaries

If f is a function on B_m, we write ${}^{vv}f$ for the function on $t(B_m)$ obtained by forming the composition of $\pi : t(B_m) \to B_m$ and ${}^{v}f = f \circ \pi_1$, so that ${}^{vv}f = {}^{v}f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi$.

Consequently,

$$ {}^{vv}f(x^a, x^\alpha, x^{\bar{\alpha}}) = f(x^a) \tag{5} $$

is provided by the ${}^{vv}f$ – vertical lift of the function $f \in \Im_0^0(B_m)$ to $t(B_m)$ .It should be observed that along every fiber of $\pi : t(B_m) \to B_m$, the value ${}^{vv}f$ stays constant. If $f = f(x^a, x^\alpha)$ is a function in M_n, on the other hand, we write ${}^{cc}f$ for the function in $t(B_m)$ defined by

$$ {}^{cc}f = y(df) = y^{\bar{\beta}}\partial_\beta f = y^{\beta}\partial_\beta f \tag{6} $$

and name the complete lift ${}^{cc}f$ of the function f [14]. ${}^{HH}f = {}^{cc}f - \nabla_\gamma f$ determines the ${}^{HH}f$ — horizontal lift of the function f to $t(B_m)$ where $\nabla_\gamma f = \gamma \nabla f$. Let $X \in \Im_0^1(B_m)$ be ($X = X^\alpha \partial_\alpha$). By using (3), we have the usual law ${}^{vv}X' = \bar{A}({}^{vv}X)$ of coordinate transformation of a vector field on $t(B_m)$ when

$$ {}^{vv}X : \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix}, \tag{7} $$

is put in. The $(1,0)-$tensor field ${}^{vv}X$ is called the vertical lift of X to semi-tangent bundle [20]. Let $\omega \in \Im_1^0(B_m)$, i.e. $\omega = \omega_\alpha dx^\alpha$. On putting

$$ {}^{vv}\omega : (0, \omega_\alpha, 0), \tag{8} $$

from (3), we easily see that ${}^{vv}\omega = \overline{A}^{vv}\omega'$. The vertical lift of ω to $t(B_m)$ is the name of the (0,1)-tensor field ${}^{vv}\omega$ [20]. The complete lift ${}^{cc}\omega \in \Im_1^0(t(B_m))$ of $\omega \in \Im_1^0(B_m)$ with the components ω_α in B_m has the following components:

$$ {}^{cc}\omega : \left(0, y^\varepsilon \partial_\varepsilon \omega_\alpha, \omega_\alpha\right) \tag{9} $$

relative to the induced coordinates in the semi-tangent bundle [20]. Let ω be a covector field on B_m with an affine connection ∇. Then the components of the ${}^{HH}\omega -$ horizontal lift of ω have the form

$$ {}^{HH}\omega = {}^{cc}\omega - \nabla_\gamma \omega $$

in $t(B_m)$, where $\nabla_\gamma \omega = \gamma \nabla \omega$. The horizontal lift ${}^{HH}\omega \in \Im_1^0(t(B_m))$ of ω has the following components:

$$ {}^{HH}\omega : \left(0, \Gamma_\alpha^\varepsilon \omega_\varepsilon, \omega_\alpha\right) $$

relative to the induced coordinates in $t(B_m)$. Now, consider that there is given a $(p,q)-$tensor field S whose local expression is

$$ S = S_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \dots \otimes dx^{\beta_q} $$

in base manifold B_m with $\nabla -$ affine connection and a $\nabla_\gamma S -$ tensor field defined by

$$ \nabla_\gamma S = y^\varepsilon \nabla_\varepsilon S_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \dots \otimes dx^{\beta_q} $$

relative to the induced coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ in $\pi^{-1}(U)$ in the semi-tangent bundle. Additionally, we define a $\nabla_X S -$ tensor field in $\pi^{-1}(U)$ by

$$ \nabla_X S = \left(X^\varepsilon S_{\varepsilon\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}\right) \frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \dots \otimes dx^{\beta_q} $$

and a $\gamma S -$ tensor field in $\pi^{-1}(U)$ by

$$ \nabla S = \left(y^\varepsilon S_{\varepsilon\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}\right) \frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \dots \otimes dx^{\beta_q} $$

relative to the induced coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$, U being an arbitrary coordinate neighborhood in B_m.

Next, we obtain $\nabla_X S = {}^{vv}(S_X)$ for any $X \in \Im_0^1(B_m)$ and $S \in \Im_s^0(B_m)$ or $S \in \Im_s^1(B_m)$, where $S_X \in \Im_{s-1}^0(B_m)$ or $\Im_{s-1}^1(B_m)$. The ${}^{HH}S-$horizontal lift of $(p,q)-$tensor field S in base manifold B_m to $t(B_m)$ has the following equation:

$$^{HH}S = {}^{cc}S - \nabla_\gamma S .$$

Assuming P, $Q \in t(B_m)$, we obtain

$$\nabla_\gamma (P \otimes Q) = {}^{vv}P \otimes (\nabla_\gamma Q) + (\nabla_\gamma P) \otimes {}^{vv}Q \text{ and } {}^{HH}(P \otimes Q) = {}^{HH}P \otimes {}^{vv}Q + {}^{vv}P \otimes {}^{HH}Q .$$

Assume $\widetilde{X} \in \Im_0^1(M_n)$ is a projectable $(1,0)-$tensor field with projection $X = X^\alpha (x^\alpha)\partial_\alpha$, i.e. $\widetilde{X} = \widetilde{X}^a (x^a, x^\alpha)\partial_a + X^\alpha (x^\alpha)\partial_\alpha$. Now, take into account $\widetilde{X} \in \Im_0^1(M_n)$, in that case complete lift ${}^{cc}\widetilde{X}$ has components of the form [14]:

$$^{cc}\widetilde{X} : \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix} \tag{10}$$

relative to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$ on the semi-tangent bundle $t(B_m)$. For an arbitrary affinor field $F \in \Im_1^1(B_m)$, if (3) is taken into consideration, we may demonstrate that $(\gamma F)' = \overline{A}(\gamma F)$, where γF is a (1,0)-tensor field defined by [9]:

$$\gamma F : \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon F_\varepsilon^\alpha \end{pmatrix} \tag{11}$$

relative to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$. For each projectable vector field $\widetilde{X} \in \Im_0^1(M_n)$ [20], we well-know that the ${}^{HH}\widetilde{X}-$horizontal lift of $\widetilde{X}$ to $t(B_m)$ (see [9]) by ${}^{HH}\widetilde{X} = {}^{cc}\widetilde{X} - \gamma(\nabla \widetilde{X})$. In the above situation, a differentiable manifold B_m has a projectable symmetric linear connection denoted by ∇. We recall that $\gamma(\nabla\widetilde{X})-$ vector field has components [9]:

$$\gamma(\nabla\widetilde{X}):\begin{pmatrix}0\\0\\y^{\varepsilon}\nabla_{\varepsilon}X^{\alpha}\end{pmatrix}$$

relative to the coordinates $(x^{a},x^{\alpha},x^{\bar{\alpha}})$ on $t(B_m)$. $\nabla_{\alpha}X^{\varepsilon}$ being the covariant derivative of X^{ε}, i.e.,

$$(\nabla_{\alpha}X^{\varepsilon})=\partial_{\alpha}X^{\varepsilon}+X^{\beta}\Gamma_{\beta}{}^{\varepsilon}{}_{\alpha}.$$

Consequently, the ${}^{HH}\widetilde{X}-$horizontal lift of $\widetilde{X}$ to $t(B_m)$ contains the following components [9]:

$${}^{HH}\widetilde{X}:\begin{pmatrix}\widetilde{X}^{a}\\X^{\alpha}\\-\Gamma^{\alpha}{}_{\beta}X^{\beta}\end{pmatrix} \tag{12}$$

relative to the coordinates $(x^{a},x^{\alpha},x^{\bar{\alpha}})$ on $t(B_m)$ where

$$\Gamma^{\alpha}{}_{\beta}=y^{\varepsilon}\Gamma_{\varepsilon}{}^{\alpha}{}_{\beta}. \tag{13}$$

Vertical lifts are given by the following relations:

$${}^{vv}(P\otimes Q)={}^{vv}P\otimes{}^{vv}Q,\quad {}^{vv}(P+R)={}^{vv}P+{}^{vv}R \tag{14}$$

to an algebraic isomorphism (unique) of the $\Im(B_m)-$tensor algebra into the $\Im(t(B_m))-$tensor algebra with respect to constant coefficients. Where P, Q and R being arbitrary elements of $t(B_m)$. For an arbitrary affinor field $F\in\Im_1^1(B_m)$, if (3) is taken into consideration, we may demonstrate that ${}^{vv}F_{J'}^{I'}=A_{I'}^{I}A_{J}^{J'}({}^{vv}F_{J'}^{I'})$, where ${}^{vv}F$ is a (1,1)-tensor field defined by [20]:

$${}^{vv}F:\begin{pmatrix}0&0&0\\0&0&0\\0&F_{\beta}^{\alpha}&0\end{pmatrix} \tag{15}$$

relative to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$. The $(1,1)-$tensor field (15) is called the vertical lift of affinor field F to $t(B_m)$ [20]. Complete lifts are given by the following relations:

$$^{cc}(P+R) = {}^{cc}P + {}^{cc}R,\ {}^{cc}(P \otimes Q) = {}^{cc}P \otimes {}^{vv}Q + {}^{vv}P \otimes {}^{cc}Q, \tag{16}$$

to an algebraic isomorphism (unique) of the $\Im(B_m)-$tensor algebra into the $\Im(t(B_m))-$ tensor algebra with respect to constant coefficients. Where P, Q and R being arbitrary elements of $t(B_m)$. For an arbitrary projectable affinor field $\widetilde{F} \in \Im_1^1(M_n)$ [20] with projection $F = F_\beta^\alpha(x^\alpha)\partial_\alpha \otimes dx^\beta$ i.e. $\widetilde{F}$ has components

$$\widetilde{F}: \begin{pmatrix} \widetilde{F}_b^a(x^a, x^\alpha) & \widetilde{F}_\beta^a(x^a, x^\alpha) \\ 0 & \widetilde{F}_\beta^\alpha(x^\alpha) \end{pmatrix}$$

relative to the coordinates (x^a, x^α). If (3) is taken into consideration, we may demonstrate that ${}^{cc}\widetilde{F}_{J'}^{I'} = A_{I'}^{I} A_J^{J'}({}^{cc}\widetilde{F}_{J'}^{I'})$, where ${}^{cc}\widetilde{F}$ is a (1,1)-tensor field defined by [20]:

$$^{cc}\widetilde{F}: \begin{pmatrix} \widetilde{F}_b^a & \widetilde{F}_\beta^a & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & y^\varepsilon \partial_\varepsilon F_\beta^\alpha & F_\beta^\alpha \end{pmatrix}, \tag{17}$$

relative to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$. The $(1,1)-$tensor field (17) is called the complete lift of affinor field F to semi-tangent bundle $t(B_m)$ [20]. We will now give below some important equations that we will use.

Lemma 1. Let $\widetilde{X}, \widetilde{Y}$ and $\widetilde{F}$ be projectable vector and $(1,1)-$tensor fields on M_n with projections X, Y and F on base manifold B_m, respectively. If $f \in \Im_0^0(B_m)$, $\omega \in \Im_1^0(B_m)$ and $I = id_{B_m}$, then [17], [18] yerine [19], [20]:

(i) ${}^{vv}\omega\, {}^{vv}X = 0$, ***(ii)*** ${}^{cc}\widetilde{X}\, {}^{vv}f = {}^{vv}(Xf)$, ***(iii)*** ${}^{vv}\omega\left({}^{cc}\widetilde{X}\right) = {}^{vv}(\omega(X))$, ***(iv)*** ${}^{vv}F\, {}^{cc}\widetilde{X} = {}^{vv}(FX)$,

(v) ${}^{vv}X\,{}^{cc}f = {}^{vv}(Xf)$, ***(vi)*** ${}^{cc}\left(\widetilde{fX}\right) = {}^{cc}f\,{}^{vv}X + {}^{vv}f\,{}^{cc}\widetilde{X} = {}^{cc}\left(\widetilde{Xf}\right)$, ***(vii)*** ${}^{vv}I\,{}^{cc}\widetilde{X} = {}^{vv}X$, ***(viii)*** ${}^{cc}\tilde{I} = \tilde{I}$,

(ix) $\left[{}^{vv}X, {}^{cc}\widetilde{Y}\right] = {}^{vv}[X,Y]$, ***(x)*** ${}^{cc}\omega\left({}^{vv}X\right) = {}^{vv}\left(\omega(X)\right)$, ***(xi)*** ${}^{vv}F\,{}^{vv}X = 0$, ***(xii)*** ${}^{cc}\widetilde{F}\,{}^{vv}X = {}^{vv}(FX)$

(xiii) ${}^{cc}\widetilde{X}\,{}^{cc}f = {}^{cc}(Xf)$, ***(xiv)*** ${}^{cc}\omega\left({}^{cc}\widetilde{X}\right) = {}^{cc}(\omega X)$, ***(xv)*** ${}^{cc}\left(\widetilde{FX}\right) = {}^{cc}\widetilde{F}\,{}^{cc}\widetilde{X}$, ***(xvi)*** ${}^{vv}(fX) = {}^{vv}f\,{}^{vv}X$,

(xvii) ${}^{vv}I\,{}^{vv}X = 0$, ***(xviii)*** $\left[{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}\right] = {}^{cc}\left[\widetilde{X,Y}\right]$, ***(xix)*** ${}^{vv}(f\omega) = {}^{vv}f\,{}^{vv}\omega$, ***(xx)*** ${}^{vv}X\,{}^{vv}f = 0$,

(xxi) $\left[{}^{vv}X, {}^{vv}Y\right] = 0$.

3. Main Results

Let an $m-$dimensional differentiable manifold B_m $(m = 2k+1,\ k \geq 0)$ be endowed with a projectable $(1,1)-$tensor field $\tilde{\varphi} \in \Im_1^1(M_n)$ [20] with projection $\varphi = \varphi_\beta^\alpha\left(x^\alpha\right)\partial_\alpha \otimes dx^\beta$ i.e., a projectable $(1,0)-$tensor field $\tilde{\xi} \in \Im_0^1(M_n)$ with projection $\xi = \xi^\alpha\left(x^\alpha\right)\partial_\alpha$ i.e. $\tilde{\xi} = \tilde{\xi}^a(x^\alpha, x^\alpha)\partial_a + \xi^\alpha\left(x^\alpha\right)\partial_\alpha$ [20], a 1-form η, I be an idendity and let them satisfy

$$\tilde{\varphi}^2 = -I + \eta \otimes \tilde{\xi}, \qquad \tilde{\varphi}\left(\tilde{\xi}\right) = 0, \qquad \eta \circ \tilde{\varphi} = 0, \qquad \eta\left(\tilde{\xi}\right) = 1. \tag{18}$$

Then $\left(\tilde{\varphi}, \tilde{\xi}, \eta\right)$ define almost contact structure on B_m (see, for example [6], [11], [12], [16], [19], [23]). Taking account of (18) we obtain

$$\left({}^{cc}\tilde{\varphi}\right)^2 = -I + {}^{vv}\eta \otimes {}^{cc}\tilde{\xi} + {}^{cc}\eta \otimes {}^{vv}\xi, \tag{19}$$

$${}^{cc}\tilde{\varphi}\,{}^{vv}\xi = 0,\quad {}^{vv}\eta \circ {}^{cc}\tilde{\varphi} = 0,\quad {}^{vv}\eta \circ {}^{cc}\tilde{\varphi} = 0,$$

$${}^{cc}\eta \circ {}^{cc}\tilde{\varphi} = 0,\quad {}^{vv}\eta\left({}^{vv}\xi\right) = 0,\quad {}^{vv}\eta\left({}^{cc}\tilde{\xi}\right) = 1,\quad {}^{cc}\eta\left({}^{vv}\xi\right) = 1,\quad {}^{cc}\eta\left({}^{cc}\tilde{\xi}\right) = 0,$$ [20]

Using (18) and (19) we define a (1,1) tensor field $\tilde{J}$ on $t(B_m)$ by

$$\tilde{J} = {}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta \tag{20}$$

Then it is easy to show that $\tilde{J}^2\,{}^{vv}X = -{}^{vv}X$ and $\tilde{J}^2\,{}^{cc}\widetilde{X} = -{}^{cc}\widetilde{X}$, which give that $\tilde{J}$ is an almost contact structure on $t(B_m)$. We get from (20)

$$\widetilde{J}\,{}^{vv}X = {}^{vv}(\varphi X) + {}^{vv}(\eta(X))\,{}^{cc}\widetilde{\xi},$$
$$\widetilde{J}\,{}^{cc}\widetilde{X} = {}^{cc}\widetilde{(\varphi X)} - {}^{vv}(\eta(X))\,{}^{vv}\xi + {}^{cc}(\eta(X))\,{}^{cc}\widetilde{\xi}$$

for any $\widetilde{X} \in \Im_0^1(M_n)$.

Let an $m-$dimensional differentiable manifold B_m $(m = 2k+1,\ k \geq 0)$ be endowed with a projectable $(1,1)-$tensor field $\widetilde{\varphi} \in \Im_1^1(M_n)$ [20] with projection $\varphi = \varphi_\beta^\alpha(x^\alpha)\partial_\alpha \otimes dx^\beta$ i.e., a projectable $(1,0)-$tensor field $\widetilde{\xi} \in \Im_0^1(M_n)$ with projection $\xi = \xi^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{\xi} = \widetilde{\xi}^a(x^\alpha, x^\alpha)\partial_a + \xi^\alpha(x^\alpha)\partial_\alpha$ [20], a 1-form η, I be an idendity and let them satisfy

$$\widetilde{\varphi}^2 = I - \eta \otimes \widetilde{\xi}, \quad \widetilde{\varphi}(\widetilde{\xi}) = 0, \quad \eta \circ \widetilde{\varphi} = 0, \quad \eta(\widetilde{\xi}) = 1. \tag{21}$$

Then $(\widetilde{\varphi}, \widetilde{\xi}, \eta)$ define almost paracontact structure on B_m (see, for example [2], [3], [6], [13], [21]). We continue taking complete and vertical lifts from (21):

$$\left({}^{cc}\widetilde{\varphi}\right)^2 = I - {}^{vv}\eta \otimes {}^{cc}\widetilde{\xi} - {}^{cc}\eta \otimes {}^{vv}\xi, \tag{22}$$

$${}^{cc}\widetilde{\varphi}\,{}^{vv}\xi = 0,\ {}^{cc}\widetilde{\varphi}\,{}^{cc}\widetilde{\xi} = 0,\ {}^{vv}\eta\,{}^{cc}\widetilde{\varphi} = 0,$$
$${}^{cc}\eta \circ {}^{cc}\widetilde{\varphi} = 0,\ {}^{vv}\eta({}^{vv}\xi) = 0,\ {}^{vv}\eta({}^{cc}\widetilde{\xi}) = 1,\ {}^{cc}\eta({}^{vv}\xi) = 1,\ {}^{cc}\eta({}^{cc}\widetilde{\xi}) = 0,$$ [20].

Using (21) and (22), we define a $(1,1)$ tensor field $\widetilde{J}$ on $t(B_m)$ by

$$\widetilde{J} = {}^{cc}\widetilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta - {}^{cc}\widetilde{\xi} \otimes {}^{cc}\eta \tag{23}$$

Then it is easy to show that $\widetilde{J}^2\,{}^{vv}X = {}^{vv}X$ and $\widetilde{J}^2\,{}^{cc}\widetilde{X} = {}^{cc}\widetilde{X}$, which give that $\widetilde{J}$ is an almost product structure on $t(B_m)$. We get from (23)

$$\widetilde{J}\,{}^{vv}X = {}^{vv}(\varphi X) - {}^{vv}(\eta(X))\,{}^{cc}\widetilde{\xi}, \quad \widetilde{J}\,{}^{cc}\widetilde{X} = {}^{cc}(\widetilde{\varphi X}) - {}^{vv}(\eta(X))\,{}^{vv}\xi - {}^{cc}(\eta(X))\,{}^{cc}\widetilde{\xi},$$

for any $\widetilde{X} \in \Im_0^1(M_n)$, (see, for example [6]).

Theorem 1. In accordance with (23), we have the following for the L_X- operator Lie derivation with respect to $\widetilde{J} \in \Im_1^1(t(B_m))$ and $\eta(Y) = 0$:

(i) $\left(L_{{}^{vv}X}\widetilde{J}\right){}^{vv}Y = 0,$ ***(ii)*** $\left(L_{{}^{vv}X}\widetilde{J}\right){}^{cc}Y = {}^{vv}\left((L_X\varphi)Y\right) - {}^{vv}\left((L_X\eta)Y\right){}^{cc}\widetilde{\xi},$

(iii) $\left(L_{{}^{cc}\widetilde{X}}\tilde{J}\right){}^{vv}Y = {}^{vv}\left(\left(L_X\varphi\right)Y\right) - {}^{vv}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$,

(iv) $\left(L_{{}^{cc}\widetilde{X}}\tilde{J}\right){}^{cc}Y = {}^{cc}\widetilde{\left(\left(L_X\varphi\right)Y\right)} - {}^{vv}\left(\left(L_X\eta\right)Y\right){}^{vv}\xi - {}^{cc}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$,

where $\widetilde{X},\widetilde{Y},\tilde{\xi} \in \Im_0^1\left(M_n\right)$ are projectable $\left(1,0\right)-$tensor fields, $\eta \in \Im_1^0\left(M_n\right)$ is a $1-$form and $\tilde{\varphi} \in \Im_1^1\left(M_n\right)$ is a projectable $\left(1,1\right)-$tensor field.

Proof 1. For $\tilde{J} = {}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta - {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta$ and $\eta\left(Y\right) = 0$, we obtain

(i) $$\left(L_{{}^{vv}X}\tilde{J}\right){}^{vv}Y = L_{{}^{vv}X}\left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta - {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right){}^{vv}Y - \left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta - {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right)\underbrace{L_{{}^{vv}X}{}^{vv}Y}_{0}$$

$$= \underbrace{L_{{}^{vv}X}{}^{vv}\left(\varphi Y\right)}_{0} - L_{{}^{vv}X}{}^{vv}\xi\left(\underbrace{{}^{vv}\eta\,{}^{vv}\left(Y\right)}_{0}\right) - L_{{}^{vv}X}{}^{cc}\tilde{\xi}\;{}^{vv}\left(\underbrace{\eta\left(Y\right)}_{0}\right) = 0$$

(ii) $$\left(L_{{}^{vv}X}\tilde{J}\right){}^{cc}\widetilde{Y} = L_{{}^{vv}X}\left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta - {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right){}^{cc}\widetilde{Y} - \left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta - {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right)L_{{}^{vv}X}{}^{cc}\widetilde{Y}$$

$$= \underbrace{L_{{}^{vv}X}{}^{cc}\tilde{\varphi}\,{}^{cc}\widetilde{Y}}_{\underbrace{\left(L_{{}^{vv}X}{}^{cc}\tilde{\varphi}\right){}^{cc}\widetilde{Y}}_{{}^{vv}\left(\left(L_X\varphi\right)Y\right)} + \underbrace{{}^{cc}\tilde{\varphi}\left(L_{{}^{vv}X}{}^{cc}\widetilde{Y}\right)}_{{}^{cc}\tilde{\varphi}\,{}^{vv}\left(L_XY\right)}} - L_{{}^{vv}X}{}^{vv}\xi\;{}^{vv}\left(\underbrace{\eta\left(Y\right)}_{0}\right) - \underbrace{L_{{}^{vv}X}{}^{cc}\tilde{\xi}\,{}^{cc}\left(\eta\left(Y\right)\right)}_{{}^{cc}\left(\eta\left(Y\right)\right)L_{{}^{vv}X}{}^{cc}\tilde{\xi} + {}^{cc}\tilde{\xi}\left(L_{{}^{vv}X}{}^{cc}\left(\eta\left(Y\right)\right)\right)} - {}^{cc}\tilde{\varphi}\,{}^{vv}\left(L_XY\right)$$

$$+ \underbrace{{}^{vv}\eta\,{}^{vv}\left(L_XY\right)}_{0}{}^{vv}\xi + \underbrace{{}^{vv}\left(\eta\left(L_XY\right)\right)}_{{}^{vv}\left(L_X\eta\left(Y\right)\right) - {}^{vv}\left(\left(L_X\eta\right)Y\right)}{}^{cc}\tilde{\xi}$$

$$= {}^{vv}\left(\left(L_X\varphi\right)Y\right) + {}^{cc}\tilde{\varphi}\,{}^{vv}\left(L_XY\right) - \underbrace{{}^{cc}\left(\eta\left(Y\right)\right)}_{0}L_{{}^{vv}X}{}^{cc}\tilde{\xi} - {}^{cc}\tilde{\xi}\underbrace{\left(L_{{}^{vv}X}{}^{cc}\left(\eta\left(Y\right)\right)\right)}_{{}^{vv}\left(L_X\eta\left(Y\right)\right)}$$

$$- {}^{cc}\tilde{\varphi}\,{}^{vv}\left(L_XY\right) + {}^{vv}\left(L_X\eta\left(Y\right)\right){}^{cc}\tilde{\xi} - {}^{vv}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$$

$$= {}^{vv}\left(\left(L_X\varphi\right)Y\right) - {}^{vv}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi},$$

(iii) $$\left(L_{{}^{cc}X}\tilde{J}\right){}^{vv}Y = L_{{}^{cc}\widetilde{X}}\left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta - {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right){}^{vv}Y - \left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta - {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right)L_{{}^{cc}\widetilde{X}}{}^{vv}Y$$

$$= L_{{}^{cc}\widetilde{X}}{}^{cc}\tilde{\varphi}\,{}^{vv}Y - L_{{}^{cc}\widetilde{X}}\left({}^{vv}\eta\,{}^{vv}\left(Y\right)\right){}^{vv}\xi - L_{{}^{cc}\widetilde{X}}{}^{vv}\left(\eta\left(Y\right)\right){}^{cc}\tilde{\xi} - {}^{cc}\tilde{\varphi}L_{{}^{cc}\widetilde{X}}{}^{vv}Y$$

$$+ {}^{vv}\eta\,{}^{vv}\left(L_XY\right){}^{vv}\xi + {}^{vv}\left(\eta\left(L_XY\right)\right){}^{cc}\tilde{\xi}$$

$$= \underbrace{L_{{}^{cc}\widetilde{X}}{}^{cc}\tilde{\varphi}\,{}^{vv}Y}_{\underbrace{{}^{vv}\left(L_X\varphi Y\right)}_{{}^{vv}\left(\left(L_X\varphi\right)Y\right) + {}^{vv}\left(\varphi L_XY\right)}} - L_{{}^{cc}\widetilde{X}}{}^{vv}\xi\left(\underbrace{{}^{vv}\eta\,{}^{vv}\left(Y\right)}_{0}\right) - \underbrace{L_{{}^{cc}\widetilde{X}}{}^{cc}\tilde{\xi}\,{}^{vv}\left(\eta\left(Y\right)\right)}_{{}^{vv}\left(\eta\left(Y\right)\right)\left(L_{{}^{cc}\widetilde{X}}{}^{cc}\tilde{\xi}\right) + {}^{cc}\tilde{\xi}\left(L_{{}^{cc}\widetilde{X}}{}^{vv}\left(\eta\left(Y\right)\right)\right)} - \underbrace{{}^{cc}\tilde{\varphi}L_{{}^{cc}\widetilde{X}}{}^{vv}Y}_{{}^{vv}\left(\varphi L_XY\right)}$$

$$+\underbrace{{}^{vv}\eta\,{}^{vv}\left(L_X Y\right)}_{0}{}^{vv}\xi+\underbrace{{}^{vv}\left(\eta\left(L_X Y\right)\right)}_{{}^{vv}\left(L_X\eta(Y)\right)-{}^{vv}\left(\left(L_X\eta\right)Y\right)}{}^{cc}\tilde{\xi}$$

$$={}^{vv}\left(\left(L_X\varphi\right)Y\right)+{}^{vv}\left(\varphi L_X Y\right)-{}^{vv}\left(\eta(Y)\right)\underbrace{\left(L_{{}^{cc}\tilde{X}}\,{}^{cc}\tilde{\xi}\right)}_{{}^{cc}\widetilde{\left(L_X\xi\right)}}-{}^{cc}\tilde{\xi}\left(\underbrace{L_{{}^{cc}\tilde{X}}\,{}^{vv}\left(\eta(Y)\right)}_{{}^{vv}\left(L_X\eta(Y)\right)}\right)$$

$$-{}^{vv}\left(\varphi L_X Y\right)+{}^{vv}\left(L_X\eta\left(Y\right)\right){}^{cc}\tilde{\xi}-{}^{vv}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$$

$$={}^{vv}\left(\left(L_X\varphi\right)Y\right)-{}^{vv}\left(\underbrace{\eta(Y)}_{0}\right){}^{cc}\widetilde{\left(L_X\xi\right)}-{}^{vv}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$$

$$={}^{vv}\left(\left(L_X\varphi\right)Y\right)-{}^{vv}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi},$$

(iv) $$\left(L_{{}^{cc}\tilde{X}}\tilde{J}\right){}^{cc}\tilde{Y}=L_{{}^{cc}\tilde{X}}\left({}^{cc}\tilde{\varphi}-{}^{vv}\xi\otimes{}^{vv}\eta-{}^{cc}\tilde{\xi}\otimes{}^{cc}\eta\right){}^{cc}\tilde{Y}-\left({}^{cc}\tilde{\varphi}-{}^{vv}\xi\otimes{}^{vv}\eta-{}^{cc}\tilde{\xi}\otimes{}^{cc}\eta\right)L_{{}^{cc}\tilde{X}}\,{}^{cc}\tilde{Y}$$

$$+\underbrace{{}^{vv}\eta\,{}^{cc}\left(L_X Y\right)}_{{}^{vv}\left(\eta\left(L_X Y\right)\right)}{}^{vv}\xi+\underbrace{{}^{cc}\left(\eta\left(L_X Y\right)\right)}_{{}^{cc}\left(L_X\eta(Y)\right)-{}^{cc}\left(\left(L_X\eta\right)Y\right)}{}^{cc}\tilde{\xi}$$

$$={}^{cc}\widetilde{\left(\left(L_X\varphi\right)Y\right)}+{}^{cc}\widetilde{\left(\varphi L_X Y\right)}-\underbrace{{}^{cc}\left(\eta(Y)\right)}_{0}{}^{cc}\left(L_{{}^{cc}\tilde{X}}\,{}^{cc}\tilde{\xi}\right)-{}^{cc}\tilde{\xi}\,\underbrace{{}^{cc}L_{{}^{cc}\tilde{X}}\,{}^{cc}\left(\eta(Y)\right)}_{{}^{cc}\left(L_X\eta(Y)\right)}$$

$$-{}^{cc}\widetilde{\left(\varphi L_X Y\right)}+{}^{vv}\left(\eta\left(L_X Y\right)\right){}^{vv}\xi+{}^{cc}\left(L_X\eta\left(Y\right)\right){}^{cc}\tilde{\xi}-{}^{cc}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$$

$$={}^{cc}\widetilde{\left(\left(L_X\varphi\right)Y\right)}+\underbrace{{}^{vv}\left(\eta\left(L_X Y\right)\right)}_{{}^{vv}\left(L_X\eta(Y)\right)-{}^{vv}\left(\left(L_X\eta\right)Y\right)}{}^{vv}\xi-{}^{cc}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$$

$$={}^{cc}\widetilde{\left(\left(L_X\varphi\right)Y\right)}+\underbrace{{}^{vv}\xi\,{}^{vv}\left(L_X\eta\left(Y\right)\right)}_{0}-{}^{vv}\left(\left(L_X\eta\right)Y\right){}^{vv}\xi-{}^{cc}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$$

$$={}^{cc}\widetilde{\left(\left(L_X\varphi\right)Y\right)}-{}^{vv}\left(\left(L_X\eta\right)Y\right){}^{vv}\xi-{}^{cc}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi},$$

where $\eta L_X Y=L_X\eta\left(Y\right)-\left(L_X\eta\right)Y$ and $\widetilde{\varphi Y}\in\Im_0^1\left(M_n\right)$. Thus, we get the following corollary:

Corollary 1. We obtain different results if we set $Y=\xi$ i.e., $\eta\left(\xi\right)=1$ and $\tilde{\xi}\in\Im_0^1(M_n)$ has the conditions of (21):

(i) $\left(L_{{}^{vv}X}\tilde{J}\right)^{vv}\xi = -{}^{vv}\left(L_X\xi\right)$, ***(ii)*** $\left(L_{{}^{vv}X}\tilde{J}\right)^{cc}\tilde{\xi} = {}^{vv}\left(\left(L_X\varphi\right)\xi\right) - {}^{vv}\left(\left(L_X\eta\right)\xi\right)^{cc}\tilde{\xi}$,

(iii) $\left(L_{{}^{cc}\tilde{X}}\tilde{J}\right)^{vv}\xi = {}^{vv}\left(\left(L_X\varphi\right)\xi\right) - {}^{cc}\widetilde{\left(L_X\xi\right)} - {}^{vv}\left(\left(L_X\eta\right)\xi\right)^{cc}\tilde{\xi}$,

(iv) $\left(L_{{}^{cc}\tilde{X}}\tilde{J}\right)^{cc}\tilde{\xi} = {}^{cc}\widetilde{\left(\left(L_X\varphi\right)\xi\right)} - {}^{vv}\left(L_X\xi\right) - {}^{vv}\left(\left(L_X\eta\right)\xi\right)^{vv}\xi - {}^{cc}\left(\left(L_X\eta\right)\xi\right)^{cc}\tilde{\xi}$.

Definition 1. Let X and J be respectively vector field and tensor field of type $(1,1)$ on M_n. A vector field X is called infinitesimal automorphisim with respect to almost contact structure J if there is $L_X J = 0$, where $(L_X J)(Y) = L_X(JY) - JL_X Y = \left[X, JY\right] - f\left[X, Y\right]$ for any $Y \in \Im_0^1\left(M_n\right)$ and L_X denotes the Lie derivative a long X [22].

Using Theorem 1 and Definition 1, we have:

Theorem 2. Let $\widetilde{X}, \widetilde{Y}$ and η be respectively projectable vector fields with projections X, Y on base manifold M_n and a $1-$form providing the condition $\eta\left(Y\right) = 0$ on B_m. Then, ${}^{vv}X$ and ${}^{cc}\tilde{X}$ are infinitesimal automorphisim with respect to almost paracontact structure $\widetilde{J}$ in semi-tangent bundle $t(B_m)$ defined by (23) if and only if $L_X\varphi = 0$ and $L_X\eta = 0$, where $\widetilde{\varphi} \in \Im_1^1\left(M_n\right)$.

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