Chapter 1

Lie derivatives of almost contact structure and almost paracontact structure with respect to ${}^{cc}\widetilde{X}$ and ${}^{vv}X$ on semi-tangent bundle $t(M)$

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Abstract

The main aim of this paper is to investigate Lie derivatives of almost contact structure and almost paracontact structure with respect to ${}^{cc}\widetilde{X}$ and ${}^{vv}X$ on semi-tangent bundle t(B).

In addition, this results which obtained shall be studied for some special vector fields in of almost contact structure and almost paracontact structure.

1. Introduction

Let B_m and M_n denote two differentiable manifolds of dimensions m and n respectively, let (M_n, π_1, B_m) be a differentiable bundle, and let π_1 be the submersion $\pi_1 : M_n \to B_m$. We may consider $(x^i) = (x^a, x^\alpha)$, $i = 1,...,n;\ a,b,... = 1,...,n-m\ ;\ \alpha, \beta,... = n-m+1,...,n$ as local coordinates in a neighborhood $\pi_1^{-1}(U)$. Let B_m be the base manifold and $\tilde{\pi} : T(B_m) \to B_m$ be the natural projection, and let $T(B_m)$ be the tangent bundle over B_m. In this case, let $T_p(B_m)$ represent in for the tangent

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space at a $p-$point $(p=x^a,x^\alpha)\in M_n, p=\pi_1(\tilde{p}))$ on the base manifold B_m. If $X^\alpha = dx^\alpha(X)$ are components of X in tangent space $T_p(B_m)$ with respect to the natural base $\{\partial_\alpha\}=\{\frac{\partial}{\partial x^\alpha}\}$, then we have the set of all points, $(x^a,x^\alpha,x^{\bar{\alpha}})$, $X^\alpha = x^{\bar{\alpha}} = y^\alpha$, $\bar{\alpha},\bar{\beta},...=n+1,...,n+m$, is by definition, the semi-tangent bundle $t(B_m)$ over the M_n manifold and the natural projection $\pi_2: t(B_m)\to M_n, \dim t(B_m)=n+m$. Specifically, assuming $n = m$, the semi-tangent bundle [14] $t(B_m)$ becomes a tangent bundle $T(B_m)$. Given a tangent bundle $\tilde{\pi}: T(B_m)\to B_m$ and a natural projection $\pi_1: M_n\to B_m$, the pullback bundle (for example see [4], [5], [7], [8], [9], [10], [17], [19], [20]) is given by $\pi_2: t(B_m)\to M_n$ where

$$t(B_m)=\left\{\left(\left(x^a,x^\alpha\right),x^{\bar{\alpha}}\right)\in M_n\times T_x(B_m)\middle|\pi_1\left(x^a,x^\alpha\right)=\tilde{\pi}\left(x^\alpha,x^{\bar{\alpha}}\right)\right\}.$$

The induced coordinates $\left(x^{1'},...,x^{n-m'},x^{1'},...,x^{m'}\right)$ with regard to $\pi^{-1}(U)$ will be given by

$$\begin{cases} x^{a'}=x^{a'}(x^b,x^\beta), & a,b,...=1,...,n-m \\ x^{\alpha'}=x^{\alpha'}\left(x^\beta\right), & \alpha,\beta,...=n-m+1,...,n, \end{cases} \tag{1}$$

If $(x^{i'})=(x^{a'},x^{\alpha'})$ is another coordinate chart on M_n, the Jacobian matrices of (1) is given by [14]:

$$\left(A_j^{i'}\right)=\left(\frac{\partial x^{i'}}{\partial x^j}\right)=\begin{pmatrix} \frac{\partial x^{a'}}{\partial x^b} & \frac{\partial x^{a'}}{\partial x^\beta} \\ 0 & \frac{\partial x^{\alpha'}}{\partial x^\beta} \end{pmatrix},$$

where $i,j,....=1,...,n.$ If (1) is local coordinate system on M_n, then we have the induced fibre coordinates $(x^{a'},x^{\alpha'},x^{\bar{\alpha}'})$ on the semi-tangent bundle (change of coordinates):

$$\begin{cases} x^{a'}=x^{a'}(x^b,x^\beta), & a,b,...=1,...,n-m, \\ x^{\alpha'}=x^{\alpha'}\left(x^\beta\right), & \alpha,\beta,...=n-m+1,...,n, \\ x^{\bar{\alpha}'}=\dfrac{\partial x^{\alpha'}}{\partial x^\beta}y^\beta, & \bar{\alpha},\bar{\beta},...=n+1,...,n+m, \end{cases} \tag{2}$$

The Jacobian matrices for (2) are as follows [14]:

$$\bar{A} = \left(A_J^{I'} \right) = \begin{pmatrix} \dfrac{\partial x^{a'}}{\partial x^b} & \dfrac{\partial x^{a'}}{\partial x^\beta} & 0 \\ 0 & \dfrac{\partial x^{\alpha'}}{\partial x^\beta} & 0 \\ 0 & y^\varepsilon \dfrac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\varepsilon} & \dfrac{\partial x^{\alpha'}}{\partial x^\beta} \end{pmatrix}, \qquad (3)$$

where $I, J, \ldots. = 1, \ldots, n + m.$ Then, we obtain

$$\left(A_{J'}^I \right) = \begin{pmatrix} A_{b'}^a & A_{\beta'}^a & 0 \\ 0 & A_{\beta'}^\alpha & 0 \\ 0 & A_{\beta'\varepsilon'}^\alpha y^{\varepsilon'} & A_{\beta'}^\alpha \end{pmatrix}, \qquad (4)$$

which is the Jacobian matrix of inverse (2).

This paper investigates lifts of several geometric objects (complete lifts, vertical lifts, etc.) that were previously studied in tangent bundles and their uses in semi-tangent bundles.

Much study has been done on the theory of tangent bundles [21], which is important in physics, differential geometry, and engineering. The semi-tangent bundle taken into consideration in this work specifies a pull-back bundle as opposed to the tangent bundle. In [1], [2], [3], [7], [12], [13] and [16], the almost paracontact structure and almost contact structure in tangent bundles have been studied along with some of their characteristics. The geometric properties of the semi-tangent bundle have been studied by several authors, such as [14], [19], [20], and others. It is known that projectable linear connections in semi-tangent bundles have been explored, along with some of their characteristics, in [9], [19] and [20].

2. Preliminaries

If f is a function on B_m, we write ${}^{vv}f$ for the function on $t(B_m)$ obtained by forming the composition of $\pi : t(B_m) \to B_m$ and ${}^{v}f = f \circ \pi_1$, so that ${}^{vv}f = {}^{v}f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi$.

Consequently,

$$ {}^{vv}f(x^a, x^\alpha, x^{\bar{\alpha}}) = f(x^\alpha) \tag{5} $$

is provided by the ${}^{vv}f$ – vertical lift of the function $f \in \Im_0^0(B_m)$ to $t(B_m)$ .It should be observed that along every fiber of $\pi : t(B_m) \to B_m$, the value ${}^{vv}f$ stays constant. If $f = f(x^a, x^\alpha)$ is a function in M_n, on the other hand, we write ${}^{cc}f$ for the function in $t(B_m)$ defined by

$$ {}^{cc}f = \iota(df) = x^{\bar{\beta}} \partial_\beta f = y^\beta \partial_\beta f \tag{6} $$

and name the complete lift ${}^{cc}f$ of the function f [14]. ${}^{HH}f = {}^{cc}f - \nabla_\gamma f$ determines the ${}^{HH}f$ — horizontal lift of the function f to $t(B_m)$ where $\nabla_\gamma f = \gamma \nabla f$. Let $X \in \Im_0^1(B_m)$ be ($X = X^\alpha \partial_\alpha$). By using (3), we have the usual law ${}^{vv}X' = \bar{A}({}^{vv}X)$ of coordinate transformation of a vector field on $t(B_m)$ when

$$ {}^{vv}X : \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix}, \tag{7} $$

is put in. The $(1,0)$–tensor field ${}^{vv}X$ is called the vertical lift of X to semi-tangent bundle [20]. Let $\omega \in \Im_1^0(B_m)$, i.e. $\omega = \omega_\alpha dx^\alpha$. On putting

$$ {}^{vv}\omega : (0, \omega_\alpha, 0), \tag{8} $$

from (3), we easily see that ${}^{vv}\omega = \bar{A}\,{}^{vv}\omega'$. The vertical lift of ω to $t(B_m)$ is the name of the (0,1)-tensor field ${}^{vv}\omega$ [20]. The complete lift ${}^{cc}\omega \in \Im_1^0(t(B_m))$ of $\omega \in \Im_1^0(B_m)$ with the components ω_α in B_m has the following components:

$$ {}^{cc}\omega : \left(0, y^\varepsilon \partial_\varepsilon \omega_\alpha, \omega_\alpha\right) \tag{9} $$

relative to the induced coordinates in the semi-tangent bundle [20]. Let ω be a covector field on B_m with an affine connection ∇. Then the components of the ${}^{HH}\omega-$horizontal lift of ω have the form

$${}^{HH}\omega = {}^{cc}\omega - \nabla_{\gamma}\omega$$

in $t(B_m)$, where $\nabla_{\gamma}\omega = \gamma\nabla\omega$. The horizontal lift ${}^{HH}\omega \in \Im_1^0(t(B_m))$ of ω has the following components:

$${}^{HH}\omega : \left(0, \Gamma_{\alpha}^{\varepsilon}\omega_{\varepsilon}, \omega_{\alpha}\right)$$

relative to the induced coordinates in $t(B_m)$. Now, consider that there is given a $(p,q)-$tensor field S whose local expression is

$$S = S_{\beta_1 \ldots \beta_q}^{\alpha_1 \ldots \alpha_p} \frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}$$

in base manifold B_m with $\nabla-$ affine connection and a $\nabla_{\gamma}S-$tensor field defined by

$$\nabla_{\gamma}S = y^{\varepsilon}\nabla_{\varepsilon}S_{\beta_1 \ldots \beta_q}^{\alpha_1 \ldots \alpha_p} \frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}$$

relative to the induced coordinates $(x^a, x^{\alpha}, x^{\bar{\alpha}})$ in $\pi^{-1}(U)$ in the semi-tangent bundle. Additionally, we define a $\nabla_X S-$tensor field in $\pi^{-1}(U)$ by

$$\nabla_X S = \left(X^{\varepsilon} S_{\varepsilon\beta_1 \ldots \beta_q}^{\alpha_1 \ldots \alpha_p}\right) \frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}$$

and a $\gamma S-$tensor field in $\pi^{-1}(U)$ by

$$\nabla S = \left(y^{\varepsilon} S_{\varepsilon\beta_1 \ldots \beta_q}^{\alpha_1 \ldots \alpha_p}\right) \frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}$$

relative to the induced coordinates $(x^a, x^{\alpha}, x^{\bar{\alpha}})$, U being an arbitrary coordinate neighborhood in B_m.

Next, we obtain $\nabla_X S = {}^{vv}(S_X)$ for any $X \in \Im_0^1(B_m)$ and $S \in \Im_s^0(B_m)$ or $S \in \Im_s^1(B_m)$, where $S_X \in \Im_{s-1}^0(B_m)$ or $\Im_{s-1}^1(B_m)$. The ${}^{HH}S-$horizontal lift of $(p,q)-$tensor field S in base manifold B_m to $t(B_m)$ has the following equation:

$${}^{HH}S = {}^{cc}S - \nabla_{\gamma}S\,.$$

Assuming $P, Q \in t(B_m)$, we obtain

$$\nabla_{\gamma}(P \otimes Q) = {}^{vv}P \otimes (\nabla_{\gamma}Q) + (\nabla_{\gamma}P) \otimes {}^{vv}Q \text{ and } {}^{HH}(P \otimes Q) = {}^{HH}P \otimes {}^{vv}Q + {}^{vv}P \otimes {}^{HH}Q\,.$$

Assume $\widetilde{X} \in \Im_0^1(M_n)$ is a projectable $(1,0)-$tensor field with projection $X = X^\alpha(x^\alpha)\partial_\alpha$, i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. Now, take into account $\widetilde{X} \in \Im_0^1(M_n)$, in that case complete lift ${}^{cc}\widetilde{X}$ has components of the form [14]:

$$ {}^{cc}\widetilde{X} : \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix} \tag{10} $$

relative to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$ on the semi-tangent bundle $t(B_m)$. For an arbitrary affinor field $F \in \Im_1^1(B_m)$, if (3) is taken into consideration, we may demonstrate that $(\gamma F)' = \bar{A}(\gamma F)$, where γF is a (1,0)-tensor field defined by [9]:

$$ \gamma F : \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon F_\varepsilon^\alpha \end{pmatrix} \tag{11} $$

relative to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$. For each projectable vector field $\widetilde{X} \in \Im_0^1(M_n)$ [20], we well-know that the ${}^{HH}\widetilde{X}-$horizontal lift of $\widetilde{X}$ to $t(B_m)$ (see [9]) by ${}^{HH}\widetilde{X} = {}^{cc}\widetilde{X} - \gamma(\nabla\widetilde{X})$. In the above situation, a differentiable manifold B_m has a projectable symmetric linear connection denoted by ∇. We recall that $\gamma(\nabla\widetilde{X})-$ vector field has components [9]:

$$ \gamma(\nabla\widetilde{X}) : \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon \nabla_\varepsilon X^\alpha \end{pmatrix} $$

relative to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$ on $t(B_m)$. $\nabla_\alpha X^\varepsilon$ being the covariant derivative of X^ε, i.e.,

$$ (\nabla_\alpha X^\varepsilon) = \partial_\alpha X^\varepsilon + X^\beta \Gamma_{\beta\ \alpha}^{\ \varepsilon}. $$

Consequently, the ${}^{HH}\widetilde{X}-$horizontal lift of $\widetilde{X}$ to $t(B_m)$ contains the following components [9]:

$$^{HH}\widetilde{X}:\begin{pmatrix}\widetilde{X}^{a}\\ X^{\alpha}\\ -\Gamma^{\alpha}{}_{\beta}X^{\beta}\end{pmatrix} \tag{12}$$

relative to the coordinates $(x^{a},x^{\alpha},x^{\bar{\alpha}})$ on $t(B_{m})$ where

$$\Gamma^{\alpha}{}_{\beta}=y^{\varepsilon}\Gamma_{\varepsilon}{}^{\alpha}{}_{\beta}. \tag{13}$$

Vertical lifts are given by the following relations:

$$^{vv}\left(P\otimes Q\right)={}^{vv}P\otimes{}^{vv}Q,\ {}^{vv}\left(P+R\right)={}^{vv}P+{}^{vv}R \tag{14}$$

to an algebraic isomorphism (unique) of the $\Im\left(B_{m}\right)-$tensor algebra into the $\Im\left(t(B_{m})\right)-$ tensor algebra with respect to constant coefficients. Where P, Q and R being arbitrary elements of $t(B_{m})$. For an arbitrary affinor field $F\in\Im_{1}^{1}(B_{m})$, if (3) is taken into consideration, we may demonstrate that $^{vv}F_{J'}^{I'}=A_{I'}^{I}A_{J}^{J'}({}^{vv}F_{J'}^{I'})$, where ^{vv}F is a (1,1)-tensor field defined by [20]:

$$^{vv}F:\begin{pmatrix}0&0&0\\0&0&0\\0&F_{\beta}^{\alpha}&0\end{pmatrix} \tag{15}$$

relative to the coordinates $(x^{a},x^{\alpha},x^{\bar{\alpha}})$. The $(1,1)-$tensor field (15) is called the vertical lift of affinor field F to $t(B_{m})$ [20]. Complete lifts are given by the following relations:

$$^{cc}\left(P+R\right)={}^{cc}P+{}^{cc}R,\ {}^{cc}\left(P\otimes Q\right)={}^{cc}P\otimes{}^{vv}Q+{}^{vv}P\otimes{}^{cc}Q, \tag{16}$$

to an algebraic isomorphism (unique) of the $\Im\left(B_{m}\right)-$tensor algebra into the $\Im\left(t(B_{m})\right)-$ tensor algebra with respect to constant coefficients. Where P, Q and R being arbitrary elements of $t(B_{m})$. For an arbitrary projectable affinor field $\widetilde{F}\in\Im_{1}^{1}(M_{n})$ [20] with projection $F=F_{\beta}^{\alpha}\left(x^{\alpha}\right)\partial_{\alpha}\otimes dx^{\beta}$ i.e. $\widetilde{F}$ has components

$$\widetilde{F}:\begin{pmatrix}\widetilde{F}_{b}^{a}(x^{a},x^{\alpha})&\widetilde{F}_{\beta}^{a}(x^{a},x^{\alpha})\\0&\widetilde{F}_{\beta}^{\alpha}(x^{\alpha})\end{pmatrix}$$

relative to the coordinates (x^a, x^α). If (3) is taken into consideration, we may demonstrate that ${}^{cc}\widetilde{F}_{J'}^{I'} = A_{I'}^{I} A_{J}^{J'} ({}^{cc}\widetilde{F}_{J'}^{I'})$, where ${}^{cc}\widetilde{F}$ is a (1,1)-tensor field defined by [20]:

$$ {}^{cc}\widetilde{F} : \begin{pmatrix} \widetilde{F}_b^a & \widetilde{F}_\beta^a & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & y^\varepsilon \partial_\varepsilon F_\beta^\alpha & F_\beta^\alpha \end{pmatrix}, \tag{17} $$

relative to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$. The $(1,1)-$tensor field (17) is called the complete lift of affinor field F to semi-tangent bundle $t(B_m)$ [20]. We will now give below some important equations that we will use.

Lemma 1. Let $\widetilde{X}, \widetilde{Y}$ and $\widetilde{F}$ be projectable vector and $(1,1)-$tensor fields on M_n with projections X, Y and F on base manifold B_m, respectively. If $f \in \Im_0^0(B_m)$, $\omega \in \Im_1^0(B_m)$ and $I = id_{B_m}$, then [17] ,[18] yerine [19], [20]:

(i) ${}^{vv}\omega\, {}^{vv}X = 0$, ***(ii)*** ${}^{cc}\widetilde{X}\, {}^{vv}f = {}^{vv}(Xf)$, ***(iii)*** ${}^{vv}\omega\left({}^{cc}\widetilde{X}\right) = {}^{vv}(\omega(X))$, ***(iv)*** ${}^{vv}F\, {}^{cc}\widetilde{X} = {}^{vv}(FX)$,

(v) ${}^{vv}X\, {}^{cc}f = {}^{vv}(Xf)$, ***(vi)*** ${}^{cc}\left(\widetilde{fX}\right) = {}^{cc}f\, {}^{vv}X + {}^{vv}f\, {}^{cc}\widetilde{X} = {}^{cc}\left(\widetilde{Xf}\right)$, ***(vii)*** ${}^{vv}I\, {}^{cc}\widetilde{X} = {}^{vv}X$, ***(viii)*** ${}^{cc}\tilde{I} = \tilde{I}$,

(ix) $\left[{}^{vv}X, {}^{cc}\widetilde{Y}\right] = {}^{vv}[X, Y]$, ***(x)*** ${}^{cc}\omega\left({}^{vv}X\right) = {}^{vv}(\omega(X))$, ***(xi)*** ${}^{vv}F\, {}^{vv}X = 0$, ***(xii)*** ${}^{cc}\widetilde{F}\, {}^{vv}X = {}^{vv}(FX)$

(xiii) ${}^{cc}\widetilde{X}\, {}^{cc}f = {}^{cc}(Xf)$, ***(xiv)*** ${}^{cc}\omega\left({}^{cc}\widetilde{X}\right) = {}^{cc}(\omega X)$, ***(xv)*** ${}^{cc}\left(\widetilde{FX}\right) = {}^{cc}\widetilde{F}\, {}^{cc}\widetilde{X}$, ***(xvi)*** ${}^{vv}(fX) = {}^{vv}f\, {}^{vv}X$,

(xvii) ${}^{vv}I\, {}^{vv}X = 0$, ***(xviii)*** $\left[{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}\right] = {}^{cc}\left[\widetilde{X, Y}\right]$, ***(xix)*** ${}^{vv}(f\omega) = {}^{vv}f\, {}^{vv}\omega$, ***(xx)*** ${}^{vv}X\, {}^{vv}f = 0$,

(xxi) $\left[{}^{vv}X, {}^{vv}Y\right] = 0$.

3. Main Results

Let an $m-$dimensional differentiable manifold B_m $(m = 2k+1,\ k \geq 0)$ be endowed with a projectable $(1,1)-$tensor field $\widetilde{\varphi} \in \Im_1^1(M_n)$ [20] with projection $\varphi = \varphi_\beta^\alpha(x^\alpha)\partial_\alpha \otimes dx^\beta$ i.e., a projectable $(1,0)-$tensor field

$\tilde{\xi} \in \Im_0^1(M_n)$ with projection $\xi = \xi^\alpha\left(x^\alpha\right)\partial_\alpha$ i.e. $\tilde{\xi} = \tilde{\xi}^a(x^\alpha, x^\alpha)\partial_a + \xi^\alpha\left(x^\alpha\right)\partial_\alpha$ [20], a 1-form η, I be an idendity and let them satisfy

$$\tilde{\varphi}^2 = -I + \eta \otimes \tilde{\xi}, \qquad \tilde{\varphi}\left(\tilde{\xi}\right) = 0, \qquad \eta \circ \tilde{\varphi} = 0, \qquad \eta\left(\tilde{\xi}\right) = 1. \tag{18}$$

Then $\left(\tilde{\varphi}, \tilde{\xi}, \eta\right)$ define almost contact structure on B_m (see, for example [2], [6], [11], [12], [16], [19], [23]). Taking account of (18) we obtain

$$\left({}^{cc}\tilde{\varphi}\right)^2 = -I + {}^{vv}\eta \otimes {}^{cc}\tilde{\xi} + {}^{cc}\eta \otimes {}^{vv}\xi, \tag{19}$$

$${}^{cc}\tilde{\varphi}\,{}^{vv}\xi = 0, \quad {}^{vv}\eta \circ {}^{cc}\tilde{\varphi} = 0, \quad {}^{vv}\eta \circ {}^{cc}\tilde{\varphi} = 0,$$

$${}^{cc}\eta \circ {}^{cc}\tilde{\varphi} = 0, \quad {}^{vv}\eta\left({}^{vv}\xi\right) = 0, \quad {}^{vv}\eta\left({}^{cc}\tilde{\xi}\right) = 1, \quad {}^{cc}\eta\left({}^{vv}\xi\right) = 1, \quad {}^{cc}\eta\left({}^{cc}\tilde{\xi}\right) = 0, \text{ [20]}$$

using (18) and (19) we define a (1,1) tensor field $\tilde{J}$ on $t(B_m)$ by

$$\tilde{J} = {}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta \tag{20}$$

Then it is easy to show that $\tilde{J}^2\,{}^{vv}X = -{}^{vv}X$ and $\tilde{J}^2\,{}^{cc}\tilde{X} = -{}^{cc}\tilde{X}$, which give that $\tilde{J}$ is an almost contact structure on $t(B_m)$. We get from (20)

$$\tilde{J}\,{}^{vv}X = {}^{vv}\left(\varphi X\right) + {}^{vv}\left(\eta\left(X\right)\right){}^{cc}\tilde{\xi},$$

$$\tilde{J}\,{}^{cc}\tilde{X} = {}^{cc}\widetilde{\left(\varphi X\right)} - {}^{vv}\left(\eta\left(X\right)\right){}^{vv}\xi + {}^{cc}\left(\eta\left(X\right)\right){}^{cc}\tilde{\xi}$$

for any $\tilde{X} \in \Im_0^1\left(M_n\right)$.

Theorem 1. In accordance with (20), we have the following for the $L_X -$ operator, Lie derivation with respect to $\tilde{J} \in \Im_1^1\left(t(B_m)\right)$ and $\eta(Y) = 0$:

(i) $\left(L_{{}^{vv}X}\tilde{J}\right){}^{vv}Y = 0$, *(ii)* $\left(L_{{}^{vv}X}\tilde{J}\right){}^{cc}\tilde{Y} = {}^{vv}\left(\left(L_X\varphi\right)Y\right) + {}^{vv}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$,

(iii) $\left(L_{{}^{cc}\tilde{X}}\tilde{J}\right){}^{vv}Y = {}^{vv}\left(\left(L_X\varphi\right)Y\right) + {}^{vv}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$,

(iv) $\left(L_{{}^{cc}\tilde{X}}\tilde{J}\right){}^{cc}\tilde{Y} = {}^{cc}\left(\widetilde{\left(L_X\varphi\right)Y}\right) - {}^{vv}\left(\left(L_X\eta\right)Y\right){}^{vv}\xi + {}^{cc}\left(\widetilde{\left(L_X\eta\right)Y}\right){}^{cc}\tilde{\xi}$,

where $\tilde{X}, \tilde{Y}, \tilde{\xi} \in \Im_0^1\left(M_n\right)$ are projectable $(1,0)-$tensor fields, $\eta \in \Im_1^0\left(M_n\right)$ is a $1-$form and $\tilde{\varphi} \in \Im_1^1\left(M_n\right)$ is a projectable $(1,1)-$tensor field.

Proof 1. For $\tilde{J} = {}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta$ and $\eta\left(Y\right) = 0$, we obtain

(i) $\left(L_{{}^{vv}X}\tilde{J}\right){}^{vv}Y = L_{{}^{vv}X}\left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right){}^{vv}Y - \left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right)\underbrace{L_{{}^{vv}X}{}^{vv}Y}_{0}$

$$= \underbrace{L_{^{vv}X}{}^{vv}(\varphi Y)}_{0} - L_{^{vv}X}\underbrace{\left({}^{vv}\eta\,{}^{vv}(Y)\right)}_{0}{}^{vv}\xi + L_{^{vv}X}\underbrace{{}^{vv}(\eta(Y))}_{0}{}^{cc}\tilde{\xi} = 0$$

(ii) $\left(L_{^{vv}X}\tilde{J}\right){}^{cc}\tilde{Y} = L_{^{vv}X}\left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right){}^{cc}\tilde{Y} - \left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right)L_{^{vv}X}{}^{cc}\tilde{Y}$

$$= \underbrace{L_{^{vv}X}{}^{cc}\tilde{\varphi}\,{}^{cc}\tilde{Y}}_{\underbrace{\left(L_{^{vv}X}{}^{cc}\tilde{\varphi}\right){}^{cc}\tilde{Y}}_{^{vv}((L_X\varphi)Y)} + \underbrace{{}^{cc}\tilde{\varphi}\left(L_{^{vv}X}{}^{cc}\tilde{Y}\right)}_{^{cc}\tilde{\varphi}\,^{vv}(L_XY)}} - L_{^{vv}X}\underbrace{{}^{vv}(\eta(Y))}_{0}{}^{vv}\xi + \underbrace{L_{^{vv}X}{}^{cc}\tilde{\xi}\,{}^{cc}(\eta(Y))}_{^{cc}(\eta(Y))L_{^{vv}X}{}^{cc}\tilde{\xi} + {}^{cc}\tilde{\xi}\left(L_{^{vv}X}{}^{cc}(\eta(Y))\right)} - {}^{cc}\tilde{\varphi}L_{^{vv}X}{}^{cc}\tilde{Y} +$$

$$+ \underbrace{{}^{vv}\eta\,{}^{vv}(L_XY)}_{0}{}^{vv}\xi - \underbrace{{}^{vv}(\eta(L_XY))}_{^{vv}(L_X\eta(Y)) - {}^{vv}((L_X\eta)Y)}{}^{cc}\tilde{\xi}$$

$$= \left(L_{^{vv}X}{}^{cc}\tilde{\varphi}\right){}^{cc}\tilde{Y} + {}^{cc}\tilde{\varphi}\left(L_{^{vv}X}{}^{cc}\tilde{Y}\right) - {}^{cc}\tilde{\varphi}L_{^{vv}X}{}^{cc}\tilde{Y} - {}^{vv}\left(L_X(\eta(Y))\right){}^{cc}\tilde{\xi} + {}^{vv}\left((L_X\eta)Y\right){}^{cc}\tilde{\xi}$$

$$= {}^{vv}((L_X\varphi)Y) + {}^{cc}\tilde{\varphi}\,{}^{vv}(L_XY) + \underbrace{{}^{cc}(\eta(Y))}_{0}L_{^{vv}X}{}^{cc}\tilde{\xi} + {}^{cc}\tilde{\xi}\underbrace{\left(L_{^{vv}X}{}^{cc}(\eta(Y))\right)}_{^{vv}(L_X\eta(Y))}$$

$$- {}^{cc}\tilde{\varphi}\,{}^{vv}(L_XY) - {}^{vv}\left(L_X\eta(Y)\right){}^{cc}\tilde{\xi} + {}^{vv}\left((L_X\eta)Y\right){}^{cc}\tilde{\xi} = {}^{vv}((L_X\varphi)Y) + {}^{vv}\left((L_X\eta)Y\right){}^{cc}\tilde{\xi}$$

(iii) $\left(L_{^{cc}\tilde{X}}\tilde{J}\right){}^{vv}Y = L_{^{cc}\tilde{X}}\left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right){}^{vv}Y - \left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right)L_{^{cc}\tilde{X}}{}^{vv}Y$

$$= \underbrace{L_{^{cc}\tilde{X}}{}^{cc}\tilde{\varphi}\,{}^{vv}Y}_{\underbrace{^{vv}(L_X\varphi Y)}_{^{vv}((L_X\varphi)Y) + {}^{vv}(\varphi L_XY)}} - L_{^{cc}\tilde{X}}\left(\underbrace{{}^{vv}\eta\,{}^{vv}(Y)}_{0}\right){}^{vv}\xi + \underbrace{L_{^{cc}\tilde{X}}{}^{cc}\tilde{\xi}\,{}^{vv}(\eta(Y))}_{^{vv}(\eta(Y))\left(L_{^{cc}\tilde{X}}{}^{cc}\tilde{\xi}\right) + {}^{cc}\tilde{\xi}\left(L_{^{cc}\tilde{X}}{}^{vv}(\eta(Y))\right)} - \underbrace{{}^{cc}\tilde{\varphi}L_{^{cc}\tilde{X}}{}^{vv}Y}_{^{vv}(\varphi L_XY)}$$

$$+ {}^{vv}\eta\,{}^{vv}(L_XY)\,{}^{vv}\xi - \underbrace{{}^{vv}(\eta(L_XY))}_{^{vv}(L_X\eta(Y)) - {}^{vv}((L_X\eta)Y)}{}^{cc}\tilde{\xi}$$

$$= {}^{vv}\left((L_X\varphi)Y\right) + {}^{vv}(\varphi L_XY) + {}^{vv}(\eta(Y))\underbrace{\left(L_{^{cc}\tilde{X}}{}^{cc}\tilde{\xi}\right)}_{^{cc}\widetilde{(L_X\xi)}} + {}^{cc}\tilde{\xi}\left(\underbrace{L_{^{cc}\tilde{X}}{}^{vv}(\eta(Y))}_{^{vv}(L_X\eta(Y))}\right)$$

$$- {}^{vv}(\varphi L_XY) - {}^{vv}\left(L_X\eta(Y)\right){}^{cc}\tilde{\xi} + {}^{vv}\left((L_X\eta)Y\right){}^{cc}\tilde{\xi}$$

$$= {}^{vv}\left((L_X\varphi)Y\right) + {}^{vv}\left(\underbrace{\eta(Y)}_{0}\right){}^{cc}\widetilde{(L_X\xi)} + {}^{vv}\left((L_X\eta)Y\right){}^{cc}\tilde{\xi} = {}^{vv}\left((L_X\varphi)Y\right) + {}^{vv}\left((L_X\eta)Y\right){}^{cc}\tilde{\xi}$$

(iv) $\left(L_{^{cc}\tilde{X}}\tilde{J}\right){}^{cc}\tilde{Y} = L_{^{cc}\tilde{X}}\left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right){}^{cc}\tilde{Y} - \left({}^{cc}\tilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{cc}\tilde{\xi} \otimes {}^{cc}\eta\right)L_{^{cc}\tilde{X}}{}^{cc}\tilde{Y}$

$$= \underbrace{L_{^{cc}X}{}^{cc}\tilde{\varphi}{}^{cc}\tilde{Y}}_{\underbrace{^{cc}\left(\widetilde{L_X\varphi Y}\right)}_{^{cc}\left(\widetilde{(L_X\varphi)Y}\right)+^{cc}\left(\widetilde{\varphi L_X Y}\right)}} - L_{^{cc}\tilde{X}}{}^{vv}\xi\underbrace{^{vv}\left(\eta(Y)\right)}_{0} + \underbrace{L_{^{cc}\tilde{X}}{}^{cc}\tilde{\xi}{}^{cc}\left(\eta(Y)\right)}_{^{cc}(\eta(Y))^{cc}\left(L_{^{cc}\tilde{X}}{}^{cc}\tilde{\xi}\right)+^{cc}\tilde{\xi}{}^{cc}L_{^{cc}\tilde{X}}{}^{cc}(\eta(Y))} - \underbrace{^{cc}\tilde{\varphi}L_{^{cc}\tilde{X}}{}^{cc}\tilde{Y}}_{^{cc}\left(\varphi L_X Y\right)}$$

$$+ \underbrace{^{vv}\eta{}^{cc}\left(L_X Y\right)}_{^{vv}\left(\eta(L_X Y)\right)}{}^{vv}\xi - \underbrace{^{cc}\left(\eta\left(L_X Y\right)\right)}_{^{cc}\left(L_X\eta(Y)\right)-^{cc}\left((L_X\eta)Y\right)}{}^{cc}\tilde{\xi}$$

$$= {}^{cc}\widetilde{\left(\left(L_X\varphi\right)Y\right)} + {}^{cc}\widetilde{\left(\varphi L_X Y\right)} + \underbrace{^{cc}\left(\eta(Y)\right)}_{0}{}^{cc}\left(L_{^{cc}\tilde{X}}{}^{cc}\tilde{\xi}\right) + {}^{cc}\tilde{\xi}\underbrace{^{cc}L_{^{cc}\tilde{X}}{}^{cc}\left(\eta(Y)\right)}_{^{cc}\left(L_X\eta(Y)\right)}$$

$$- {}^{cc}\widetilde{\left(\varphi L_X Y\right)} + {}^{vv}\left(\eta\left(L_X Y\right)\right){}^{vv}\xi - {}^{cc}\left(L_X\eta\left(Y\right)\right){}^{cc}\tilde{\xi} + {}^{cc}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$$

$$= {}^{cc}\widetilde{\left(\left(L_X\varphi\right)Y\right)} + \underbrace{^{vv}\left(\eta\left(L_X Y\right)\right)}_{^{vv}\left(L_X\eta(Y)\right)-^{vv}\left((L_X\eta)Y\right)}{}^{vv}\xi + {}^{cc}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$$

$$= {}^{cc}\widetilde{\left(\left(L_X\varphi\right)Y\right)} - \underbrace{^{vv}\xi{}^{vv}\left(L_X\eta\left(Y\right)\right)}_{0} - {}^{vv}\left(\left(L_X\eta\right)Y\right){}^{vv}\xi + {}^{cc}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi}$$

$$= {}^{cc}\widetilde{\left(\left(L_X\varphi\right)Y\right)} - {}^{vv}\left(\left(L_X\eta\right)Y\right){}^{vv}\xi + {}^{cc}\left(\left(L_X\eta\right)Y\right){}^{cc}\tilde{\xi},$$

Thus, we get the following corollary:

Corollary 1. We obtain different results if we set $Y = \xi$, i.e., $\eta\left(\xi\right) = 1$ and $\tilde{\xi} \in \Im_0^1(M_n)$ has the conditions of (18):

(i) $\left(L_{^{vv}X}\tilde{J}\right){}^{vv}\xi = {}^{vv}\left(L_X\xi\right)$, ***(ii)*** $\left(L_{^{vv}X}\tilde{J}\right){}^{cc}\tilde{\xi} = {}^{vv}\left(\left(L_X\varphi\right)\xi\right) + {}^{vv}\left(\left(L_X\eta\right)\xi\right){}^{cc}\tilde{\xi}$,

(iii) $\left(L_{^{cc}\tilde{X}}\tilde{J}\right){}^{vv}\xi = {}^{vv}\left(\left(L_X\varphi\right)\xi\right) + {}^{cc}\widetilde{\left(L_X\xi\right)} + {}^{vv}\left(\left(L_X\eta\right)\xi\right){}^{cc}\tilde{\xi}$,

(iv) $\left(L_{^{cc}\tilde{X}}\tilde{J}\right){}^{cc}\tilde{\xi} = {}^{cc}\widetilde{\left(\left(L_X\varphi\right)\xi\right)} - {}^{vv}\left(L_X\xi\right) - {}^{vv}\left(\left(L_X\eta\right)\xi\right){}^{vv}\xi + {}^{cc}\left(\left(L_X\eta\right)\xi\right){}^{cc}\tilde{\xi}$.

Definition 1. Let X and J be respectively vector field and tensor field of type (1,1) on M_n. A vector field X is called infinitesimal automorphisim with respect to almost contact structure J if there is $L_X J = 0$, where $(L_X J)(Y) = L_X(JY) - JL_X Y = \left[X, JY\right] - f\left[X, Y\right]$ for any $Y \in \Im_0^1\left(M_n\right)$ and L_X denotes the Lie derivative a long X [22].

Using Theorem 1 and Definition 1, we have:

Theorem 2. Let $\widetilde{X},\widetilde{Y}$ and η be respectively projectable vector fields with projections X,Y on base manifold M_n and a $1-$form providing the condition $\eta(Y)=0$ on B_m. Then, ${}^{vv}X$ and ${}^{cc}\tilde{X}$ are infinitesimal automorphisim with respect to almost contact structure $\widetilde{J}$ in semi-tangent bundle $t(B_m)$ defined by (20) if and only if $L_X\varphi=0$ and $L_X\eta=0$, where $\widetilde{\varphi}\in\Im_1^1(M_n)$.

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References

[1] Blair DE. Contact Manifolds in Riemannian Geometry. Lecture Notes in Maths, Vol. 509, New York: Springer Verlag, 1976.

[2] Çayır H, Koseogolu G. Lie derivatives of almost contact structure and almost paracontact structure with respect to ${}^{cc}X$ and ${}^{vv}X$ on tangent bundle T(M). New Tr. in Math. Sci., 4 (1) (2016), 153-159.

[3] Das Lovejoy S. Fiberings on almost r.contact manifolds. Publicationes Mathematicae, Debrecen, Hungary 1993; 43: 161-167.

[4] Husemoller D. Fibre Bundles. New York: Springer, 1994.

[5] Lawson HB, Michelsohn ML. Spin Geometry. Princeton: Princeton University Press, 1989.

[6] Omran T, Sharffuddin A, Husain SI. Lift of Structures on Manifolds. Publications de l'Institut Mathematique,, Nouvelle serie 1984; 360 (50): 93-97.

[7] Oproiu V. Some remarkable structures and connections defined on the tangent bundle. Rendiconti di Matematica 1973; 3 (6): 503-540.

[8] Ostianu N.M. Step-fibred spaces. Tr. Geom. Sem. 5, Moscow: (VINITI) 1974; 259-309.

[9] Polat M, Yıldırım F. Complete lifts of projectable linear connection to semi-tangent bundle. Honam Mathematical J., 2021; 43 (3): 483-501. https://doi.org/10.5831/HMJ.2021.43.3.483

[10] Pontryagin LS. Characteristic cycles on differentiable manifolds. Rec. Math. (Mat. Sbornik) N.S. 1947; 21 (63): 233-284.

[11] Poor WA. Differential Geometric Structures. New York: McGraw-Hill, 1981.

[12] Salimov AA. Tensor Operators and Their applications. New York: Nova Sci. Publ., 2013.

[13] Salimov AA, Cayır H. Some Notes On Almost Paracontact Structures. Comptes Rendus de l'Acedemie Bulgare Des Sciences 2013; 66 (3): 331-338.

[14] Salimov AA, Kadıoglu E. Lifts of Derivations to the Semi-tangent Bundle. Turk J. Math. 2000; 24 (3): 259-266.

[15] Salimov AA. Applications of Holomorphic Functions in Geometry. Birkhauser: Springer, 2023.

[16] Sasaki S. On The Differantial Geometry of Tangent Bundles of Riemannian Manifolds. Tohoku Math. J. 1958; 10 (3): 338-358. https://doi.org/10.2748/tmj/1178244668

[17] Steenrod N. The Topology of Fibre Bundles. Princeton: Princeton University Press., 1951.

[18] Şahin B, Akyol MA. Golden maps between Golden Riemannian manifolds and constancy of certain maps. Math. Com. 2014; 19 (2): 333–342.

[19] Vishnevskii V, Shirokov AP, Shurygin VV. Spaces over Algebras. Kazan: Kazan Gos. Univ., 1985 (in Russian).

[20] Vishnevskii VV. Integrable affinor structures and their plural interpretations. Geometry, 7.J. Math. Sci. (New York) 2002; 108 (2): 151-187.

[21] Yano K, Ishihara S. Tangent and Cotangent Bundles. New York: Marcel Dekker Inc., 1973.

[22] Vanžura J. Integrability conditions for polynomial structures. Ködai Math. Sem. Rep. 1976; 27, 42-50.