Chapter 5

Quaternion-Based Analysis of Line-Symmetric Motions in Lorentzian Space[1]

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Abstract

In the realm of geometric algebra and robotics, understanding the mathematical structures that describe motion is paramount. In Euclidean space, dual quaternions have emerged as a powerful tool for representing rigid body motions, encapsulating both rotation and translation in a compact, efficient manner. This elegance extends into the Lorentzian space, where dual split quaternions serve a similar role, adapting to the unique geometric properties of spacetime. This paper delves into the theory and application of dual split quaternions in representing line symmetric motions within Lorentz space, illustrating their utility and elegance in handling complex motion representations.

1. Introduction

Quaternions and dual quaternions have become indispensable tools in the representation and manipulation of movements within Euclidean spaces, particularly because of their robustness and efficiency in handling complex spatial transformations. These mathematical constructs are especially useful in areas such as robotics, computer graphics, aerospace, and virtual reality, where precise control over motion, rotation, translation, and screw movements is crucial. To fully appreciate the utility and the underlying principles of quaternions and dual quaternions, a deep dive into some foundational concepts and applications is essential. For detailed information, we refer the reader to (Bottema & Roth, 1990; Hanson, 2005; Ward, 1997; Altmann, 1986; Blaschke, 1960; Borel, 1908; Clifford, 1871). Let's expand on the basic notions used in the paper.

$$A = a + \varepsilon a^*$$

is a dual number where a and a^* are real numbers and

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$$\varepsilon^2 = 0, \varepsilon \neq 0, 0\varepsilon = \varepsilon 0 = 0, 1\varepsilon = \varepsilon 1 = \varepsilon, \varepsilon i = i\varepsilon.$$

$q = a_0 + a_1 i + a_2 j + a_3 k$ is a split quaternion. Here a_0, a_1, a_2 and a_3 are real numbers, and

$$\begin{aligned} i^2 &= -1,\ j^2 = k^2 = ijk = 1 \\ ij &= -ji = k,\ jk = -kj = -i,\ ki = -ik = j. \end{aligned}$$

$\hat{q} = \hat{a}_0 + \hat{a}_1 i + \hat{a}_2 j + \hat{a}_3 k$ is a dual split quaternion where $\hat{a}_0, \hat{a}_1, \hat{a}_2$ and $\hat{a}_3$ are dual numbers. Hence, $\hat{q}$ can be written as

$$\hat{q} = q + \varepsilon q^*$$

where $q = a_0 + a_1 i + a_2 j + a_3 k$ and $q^* = a_0^* + a_1^* i + a_2^* j + a_3^* k$ are split quaternions.

The dual split quaternion multiplication is defined as

$$\hat{q}_1 \hat{q}_2 = (q_1 q_2, q_1 q_2^* + q_1^* q_2)$$

where $\hat{q}_1 = q_1 + \varepsilon q_1^*$ and $\hat{q}_2 = q_2 + \varepsilon q_2^*$ are dual split quaternions. Furthermore, the division $\frac{\hat{q}_1}{\hat{q}_2}$ is

$$(\hat{q}_1/\hat{q}_2) = (q_1 + \varepsilon q_1^*)/(q_2 + \varepsilon q_2^*) = (q_1/q_2) + \varepsilon(q_1^* q_2 - q_1 q_2^*)/({q_2}^2)$$

where $\hat{q}_2 \neq 0$.

Additionally,

$$\begin{aligned} |\hat{q}|^2 = \hat{q}\overline{\hat{q}} = (q + \varepsilon q^*)(\bar{q} + \varepsilon \bar{q}^*) &= q\bar{q} + \varepsilon(q\bar{q}^* + q^*\bar{q}) \\ &= (a_0^2 + a_1^2 - a_2^2 - a_3^2) + 2\varepsilon(a_0 a_0^* + a_1 a_1^* - a_2 a_2^* - a_3 a_3^*) \\ &= (a_0^2 + 2\varepsilon a_0 a_0^*) + (a_1^2 + 2\varepsilon a_1 a_1^*) - (a_2^2 + 2\varepsilon a_2 a_2^*) \\ &- (a_3^2 + 2\varepsilon a_3 a_3^*) = \hat{a}_0^2 + \hat{a}_1^2 - \hat{a}_2^2 - \hat{a}_3^2 \end{aligned}$$

and $|\hat{q}| = \sqrt{|\hat{a}_0^2 + \hat{a}_1^2 - \hat{a}_2^2 - \hat{a}_3^2|}$ is the norm of $\hat{q}$ where $\hat{a}_0 = a_0 + \varepsilon a_0^*$, $\hat{a}_1 = a_1 + \varepsilon a_1^*$, $\hat{a}_2 = a_2 + \varepsilon a_2^*$ and $\hat{a}_3 = a_3 + \varepsilon a_3^*$ are dual numbers. If $|\hat{q}| = 1$ then $\hat{q}$ is a unit dual split quaternion. Besides, inverse of $\hat{q}$ is

$$\hat{q}^{-1} = \frac{\overline{\hat{q}}}{|\hat{q}|^2}$$

where $\overline{\hat{q}} = \hat{a}_0 - \hat{a}_1 i - \hat{a}_2 j - \hat{a}_3 k$ and $|\hat{q}| \neq 0$ (Kula & Yaylı, 2006; Akyar, 2008; Atasoy et al., 2017).

2. Preliminaries

A dual split quaternion $\hat{q} = \hat{a}_0 + \hat{a}_1 i + \hat{a}_2 j + \hat{a}_3 k = q + \varepsilon q^*$ with $q = a_0 + a_1 i + a_2 j + a_3 k$ and $q^* = a_0^* + a_1^* i + a_2^* j + a_3^* k$ split quaternion components is spacelike (if $I_q < 0$), timelike (if $I_q > 0$) or lightlike (if $I_q = 0$) where $I_q = a_0^2 + a_1^2 - a_2^2 - a_3^2$. As you can see, q is the decisive part.

2.1. Polar form for dual split quaternions

Let's take any dual split quaternion $\hat{q}$. Its polar forms are as follows.

i) If $\hat{q}$ is spacelike dual split quaternion then

$$\hat{q} = |\hat{q}|\left(\sinh\hat{\theta} + \hat{\mu}\cosh\hat{\theta}\right)$$

where $\sinh\hat{\theta} = \frac{\hat{a}_0}{|\hat{q}|}$, $\cosh\hat{\theta} = \frac{\sqrt{-\hat{a}_1^2+\hat{a}_2^2+\hat{a}_3^2}}{|\hat{q}|}$, $\hat{\theta} = \theta + \varepsilon\theta^*$ dual angle and $\hat{\mu} = \frac{\hat{a}_1 i+\hat{a}_2 j+\hat{a}_3 k}{\sqrt{-\hat{a}_1^2+\hat{a}_2^2+\hat{a}_3^2}}$ is spacelike unit vector.

ii) If $\hat{q}$ is timelike dual split quaternion with spacelike vector part then

$$\hat{q} = |\hat{q}|\left(\cosh\hat{\theta} + \hat{\mu}\sinh\hat{\theta}\right)$$

where $\cosh\hat{\theta} = \frac{\hat{a}_0}{|\hat{q}|}$, $\sinh\hat{\theta} = \frac{\sqrt{-\hat{a}_1^2+\hat{a}_2^2+\hat{a}_3^2}}{|\hat{q}|}$, $\hat{\theta} = \theta + \varepsilon\theta^*$ dual angle and $\hat{\mu} = \frac{\hat{a}_1 i+\hat{a}_2 j+\hat{a}_3 k.}{\sqrt{-\hat{a}_1^2+\hat{a}_2^2+\hat{a}_3^2}}$ is spacelike unit vector.

iii) If $\hat{q}$ is timelike dual split quaternion with timelike vector part then

$$\hat{q} = |\hat{q}|\left(\cos\hat{\theta} + \hat{\mu}\sin\hat{\theta}\right)$$

where $\cos\hat{\theta} = \frac{\hat{a}_0}{|\hat{q}|}$, $\sin\hat{\theta} = \frac{\sqrt{\hat{a}_1^2-\hat{a}_2^2-\hat{a}_3^2}}{|\hat{q}|}$, $\hat{\mu} = \frac{\hat{a}_1 i+\hat{a}_2 j+\hat{a}_3 k}{\sqrt{\hat{a}_1^2-\hat{a}_2^2-\hat{a}_3^2}}$ is spacelike unit vector and $\hat{\mu}^2 = \hat{\mu}\hat{\mu} = -1$.

iv) If $\hat{q}$ is unit dual split quaternion with lightlike dual split vector part then

$$\hat{q} = 1 + \hat{\mu}$$

where $\hat{\mu}$ is lightlike (null) dual split vector. $\hat{\theta} = \theta + \varepsilon\theta^*$ is a dual angle and it makes rotation as θ and translation as θ^* about the dual axis $\hat{\mu}$, where

$$\cos\hat{\theta} = \cos\theta - \varepsilon\theta^*\sin\theta$$

$$\sin\hat{\theta} = \sin\theta + \varepsilon\theta^* \cos\theta$$
$$\cosh\hat{\theta} = \cosh\theta + \varepsilon\theta^* \sinh\theta$$
$$\sinh\hat{\theta} = \sinh\theta + \varepsilon\theta^* \cosh\theta$$

For detailed information, see (Hanson, 2005; Akyar, 2008, Özdemir & Ergin, 2006; Atasoy et all., 2017).

2.2. Cross product and inner product in Lorentzian space

The equality

$$u \underset{L}{\times} v = \begin{vmatrix} -i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$$

is Lorentzian cross product and the equality

$$< u, v \underset{L}{>} = -x_1y_1 + x_2y_2 + x_3y_3$$

is Lorentzian inner product where $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$ are two vectors in Lorentz space.

In Lorentz space, for vectors u and v:

i) $u \times_L v$ is a spacelike vector if u and v are timelike vectors. Equalities

$$< u, v \underset{L}{>} = -\|u\|\|v\|\cosh\theta$$

and

$$\left\|u \underset{L}{\times} v\right\| = \|u\|\|v\|\sinh\theta$$

are written where θ is the hyperbolic angle between u and v.

ii) $u \times_L v$ is timelike if u and v are spacelike vectors satisfying the inequality $|< u, v >_L| < \|u\|\|v\|$. Equalities

$$< u, v \underset{L}{>} = \|u\|\|v\|\cos\theta$$

and

$$\|u \times_L v\| = \|u\|\|v\|\sin\theta$$

are written where θ is the angle between u and v.

iii) $u \times_L v$ is spacelike if u and v are spacelike vectors satisfying the inequality $|< u, v >_L| < \|u\|\|v\|$. Equalities

$$< u, v \underset{L}{>} = -\|u\|\|v\|\cosh\theta$$

and

$$\|u \times_L v\| = \|u\|\|v\|\sinh\theta$$

are written where θ is the hyperbolic angle between u and v.

iv) $u \times_L v$ is lightlike if u and v are spacelike vectors satisfying the equality $|< u, v >_L| = \|u\|\|v\|$. For detailed information, see (Inoguchi, 1998; Özdemir & Ergin, 2006; Kula & Yaylı, 2006, 2007).

3. Rigid transformation in Lorentzian space

A dual split quaternion representing a rigid transformation in Lorentz space is written as

$$g = r + \frac{1}{2}\varepsilon tr$$

where r is a split quaternion and t is a pure split quaternion. Here, r represent a rotation and $t = t_x i + t_y j + t_z k$ represent a translation. Any point in Lorentz space is represented as

$$\hat{p} = 1 + \varepsilon p$$

with the help of dual split quaternion, where p is pure split quaternion corresponding to this point. If p timelike, spacelike or lightlike respectively than $\hat{p}$ timelike, spacelike or lightlike. In Lorentz space, the action of a rigid transformation on a point is given by

$$\hat{p}' = \left(r + \frac{1}{2}\varepsilon tr\right)\hat{p}\left(\bar{r} - \frac{1}{2}\varepsilon\bar{r}t\right).$$

That is

$$\begin{aligned} \hat{p}' &= \left(r + \frac{1}{2}\varepsilon tr\right)(1 + \varepsilon p)\left(\bar{r} - \frac{1}{2}\varepsilon\bar{r}t\right) \\ &= r\bar{r} + \varepsilon\left(rp\bar{r} + \frac{1}{2}r\bar{r}t + \frac{1}{2}tr\bar{r}\right) \\ &= 1 + \varepsilon(rp\bar{r} + t). \end{aligned}$$

Suppose v is a unit vector and p is a point in Lorentz space, then a line in the direction of v and passing through the point p can be given with equality

$$l = v + \varepsilon p \times_L v$$

as in real space.

Notice that, as with the pure rotations, g and $-g$ represent the same rigid transformation in Lorentz space.

The effect of a rigid transformation g in Lorentz space is given by

$$l' = gl\bar{g}$$

on a line l. That is, if $g = r + \frac{1}{2}\varepsilon tr$ then $\bar{g} = \bar{r} - \frac{1}{2}\varepsilon\bar{r}t$ (Bottema & Roth, 1990). It can be seen that r is determinative about character of g.

Polar forms of g are as follows:

i) If r is a spacelike split quaternion then $g = r + \frac{1}{2}\varepsilon tr = \hat{a}_0 + \hat{a}_1 i + \hat{a}_2 j + \hat{a}_3 k$ is spacelike dual split quaternion. g can be written in the form

$$g = |g|(sinh\hat{\theta} + \hat{\mu}cosh\hat{\theta}) = sinh\hat{\theta} + \hat{\mu}cosh\hat{\theta}$$

where $|g| = 1$, $sinh\hat{\theta} = \frac{\hat{a}_0}{|q|}$, $\cosh\hat{\theta} = \frac{\sqrt{-\hat{a}_1^2+\hat{a}_2^2+\hat{a}_3^2}}{|\hat{q}|}$, $\hat{\theta} = \theta + \varepsilon\theta^*$ dual angle and $\hat{\mu} = \frac{\hat{a}_1 i+\hat{a}_2 j+\hat{a}_3 k}{\sqrt{-\hat{a}_1^2+\hat{a}_2^2+\hat{a}_3^2}}$ is a spacelike unit vector. g makes rotation about the dual axis $\hat{\mu}$ as 2θ and makes translation as $2\theta^*$.

Therefore, a action of $\hat{\theta}$ dual angle about such a $l = v + \varepsilon p \times_L v$ spacelike line in Lorentz space is given by the dual split quaternion

$$g = \sinh\left(\frac{\hat{\theta}}{2}\right) + \cosh\left(\frac{\hat{\theta}}{2}\right) l = (sinh\frac{\hat{\theta}}{2} + cosh\frac{\hat{\theta}}{2} v) + \varepsilon cos\,h\frac{\hat{\theta}}{2}(p \times v).$$

g is spacelike dual split quaternion with spacelike vector part.

To see this, notice that the rotational part of this transformation is simply the split quaternion

$$r = sinh\frac{\hat{\theta}}{2} + cosh\frac{\hat{\theta}}{2}(\mathrm{v_x i} + \mathrm{v_y j} + \mathrm{v_z k})$$

as above. If the line passes through the origin, that is if $p = 0$ then we are done, otherwise we can produce the rotation about the line by first translating it to the origin, rotating and then translating back. Thus, we can write that

$$g = \left(1 + \frac{1}{2}\varepsilon p\right) r \left(1 - \frac{1}{2}\varepsilon p\right) = r + \frac{1}{2}\varepsilon(pr - rp).$$

Finally, a simple computation confirms that the quaternion $\frac{1}{2}(pr - rp)$ corresponds to the vector $\cos h\frac{\hat{\theta}}{2} p \times v$.

ii) If r timelike split quaternion with spacelike vector part then

$$g = r + \frac{1}{2}\varepsilon tr = \hat{a}_0 + \hat{a}_1 i + \hat{a}_2 j + \hat{a}_3 k$$

timelike dual split quaternion with spacelike vector part. g can be written in the form

$$g = |g|(cosh\hat{\theta} + \hat{\mu}sinh\hat{\theta}) = cosh\hat{\theta} + \hat{\mu}sinh\hat{\theta}$$

where $|g| = 1,\ cosh\hat{\theta} = \frac{\hat{a}_0}{|q|},$ $\sinh\hat{\theta} = \frac{\sqrt{-\hat{a}_1^2+\hat{a}_2^2+\hat{a}_3^2}}{|\hat{q}|},$ $\hat{\theta} = \theta + \varepsilon\theta^*$ dual açı and $\hat{\mu} = \frac{(\hat{a}_1 i+\hat{a}_2 j+\hat{a}_3 k)}{\sqrt{-\hat{a}_1{}^2+\hat{a}_2{}^2+\hat{a}_3{}^2}}$ is a spacelike unit vector. g makes rotation about the dual axis $\hat{\mu}$ as 2θ and makes translation as $2\theta^*$.

Therefore, a action of $\hat{\theta}$ dual angle about such a $l = v + \varepsilon p \times_L v$ spacelike line in Lorentz space is given by the dual split quaternion

$$g = \cosh\left(\frac{\hat{\theta}}{2}\right) + sinh\left(\frac{\hat{\theta}}{2}\right) l = (cosh\frac{\hat{\theta}}{2} + sinh\frac{\hat{\theta}}{2} v) + \varepsilon sin\, h\frac{\hat{\theta}}{2}(p \times v)$$

g is timelike dual split quaternion with spacelike vector part.

To see this, notice that the rotational part of this transformation is simply the quaternion,

$$r = cosh\frac{\hat{\theta}}{2} + sinh\frac{\hat{\theta}}{2}(\mathrm{v_x i} + \mathrm{v_y j} + \mathrm{v_z k})$$

above. If the line passes through the origin, that is if $p = 0$ then we are done, otherwise we can produce the rotation about the line by first translating it to the origin, rotating and then translating back. We can write that

$$g = \left(1 + \frac{1}{2}\varepsilon p\right) r \left(1 - \frac{1}{2}\varepsilon p\right) = r + \frac{1}{2}\varepsilon(pr - rp).$$

Thus, a simple computation confirms that the quaternion $\frac{1}{2}(pr - rp)$ corresponds to the vector $\sinh\frac{\hat{\theta}}{2} p \times v$.

iii) If r timelike split quaternion with timelike vector part then $g = r + \frac{1}{2}\varepsilon tr = \hat{a}_0 + \hat{a}_1 i + \hat{a}_2 j + \hat{a}_3 k,$ timelike dual split quaternion with timelike vector part. g can be written in the form

$$g = |g|\left(\cos\hat{\theta} + \hat{\mu}\sin\hat{\theta}\right) = \cos\hat{\theta} + \hat{\mu}\sin\hat{\theta}$$

where $|g| = 1$, $cos\hat{\theta} = \frac{\hat{a}_0}{|q|}$, $\sin\hat{\theta} = \frac{\sqrt{-\hat{a}_1^2+\hat{a}_2^2+\hat{a}_3^2}}{|\hat{q}|}$, $\hat{\theta} = \theta + \varepsilon\theta^*$ dual angle and $\hat{\mu} = \frac{(\hat{a}_1 i+\hat{a}_2 j+\hat{a}_3 k)}{\sqrt{\hat{a}_1{}^2-\hat{a}_2{}^2+\hat{a}_3{}^2}}$ is a timelike unit vector. g makes rotation about the dual axis $\hat{\mu}$ as 2θ and makes translation as $2\theta^*$.

Therefore, a action of θ dual angle about such a $l = v + \varepsilon p \times_L v$ spacelike line in Lorentz space is given by the dual split quaternion

$$g = \cos\left(\frac{\hat{\theta}}{2}\right) + sin\left(\frac{\hat{\theta}}{2}\right) l = (cos\frac{\hat{\theta}}{2} + sin\frac{\hat{\theta}}{2} v) + \varepsilon sin\ \frac{\hat{\theta}}{2}(p \times v)$$

g is timelike dual split quaternion with timelike vector part.

To see this, notice that the rotational part of this transformation is simply the quaternion, $r = \cos\frac{\hat{\theta}}{2} + \sin\frac{\hat{\theta}}{2}\left(v_x i + v_y j + v_z k\right)$ as above. If the line passes through the origin, that is if $p = 0$ then we are done, otherwise we can produce the rotation about the line by first translating it to the origin, rotating and then translating back. We can write that

$$g = \left(1 + \frac{1}{2}\varepsilon p\right) r \left(1 - \frac{1}{2}\varepsilon p\right) = r + \frac{1}{2}\varepsilon(pr - rp).$$

So, a simple computation confirms that the quaternion $\frac{1}{2}(pr - rp)$ corresponds to the vector $\sin\frac{\hat{\theta}}{2} p \times_L v$.

iv) If r lightlike split quaternion then $g = r + \frac{1}{2}\varepsilon tr = \hat{a}_0 + \hat{a}_1 i + \hat{a}_2 j + \hat{a}_3 k$, lightlike dual split quaternion. g can be written in the form

$$g = 1 + \hat{u}$$

where $\hat{u}$ is a lightlike (null) dual split vector. g doesn't makes rotation and translation (Selig & Husty, 2011; Hanson, 2005; Atasoy et al., 2017).

Notice that not all dual split quaternions represent rigid transformations. A dual split quaternion g is a rigid transformation where

$$g\bar{g} = 1.$$

This is easily checked using the form $g = r + (1/2)\varepsilon tr$ given above and remembering that the rotation r satisfies $r\bar{r} = 1$ and since the translation t is a pure split quaternion $\bar{t} = -t$ (Selig & Husty, 2011).

4. Half-turn in Lorentzian space

A half-turn is a rotation by π radians angle about some line in Euclidean space. In Lorentz space, half-turns can be represented by dual split quaternions of the form $= (\hat{a}_1 i + \hat{a}_2 j + \hat{a}_3 k) + \varepsilon(\hat{c}_1 i + \hat{c}_2 j + \hat{c}_3 k)$. They can be thought of as reflections in the line.

The actions of the group by conjugation and adjoint preserve the set of half-turns. That is, for any group element h and half-turn l the conjugation $hl\bar{h} = l'$is another half-turn. To see this notice that these lines are the dual equivalent of the pure split quaternions, that is $\bar{l} = -l$ for half-turns. Moreover, in Lorentz space, the half-turns are the only dual split quaternions that satisfy this relation. Now the split quaternion conjugate of $hl\bar{h} = l'$ is

$$\left(\overline{l'}\right) = \left(\overline{hl\bar{h}}\right) = \left(\bar{\bar{h}}\right)\bar{l}\bar{h} = -hl\bar{h} = -l'.$$

A rigid transformation g can be written as the product of two half-turns. For example, consider a finite screw motion about the z-axis in Lorentz space, this can be written as the timelike dual split quaternion with spacelike vector part

$$g = \cosh\left(\frac{\hat{\theta}}{2}\right) + \hat{\mu}\sinh\left(\frac{\hat{\theta}}{2}\right) = \cosh\left(\frac{\theta}{2} + \varepsilon\frac{d}{2}\right) + ksinh\left(\frac{\theta}{2} + \varepsilon\frac{d}{2}\right)$$

$$= cosh\frac{\theta}{2} + \varepsilon\frac{d}{2}sinh\frac{\theta}{2} + k\left(sinh\frac{\theta}{2} + \varepsilon\frac{d}{2}cosh\frac{\theta}{2}\right)$$

$$= \left(cosh\frac{\theta}{2} + sinh\frac{\theta}{2}k\right) + \varepsilon\left(\frac{d}{2}sin\,h\frac{\theta}{2} + \frac{d}{2}cosh\frac{\theta}{2}k\right)$$

where $\hat{\theta} = \theta + \varepsilon d,\ \hat{\mu} = k + \varepsilon(0 \times_L k) = k,\ \cosh\hat{\theta} = \cosh\theta + \varepsilon\theta^*\sinh\theta,$ and $\sinh\hat{\theta} = \sinh\theta + \varepsilon\theta^*\cosh\theta.$

It is easy to see that this can be decomposed as $g = l_1 l_2$ where the two half-turns are,

$$l_1 = i$$

and

$$l_2 = \left(-\cosh\frac{\theta}{2}i + \sinh\frac{\theta}{2}j\right) + \varepsilon\left(\frac{-d}{2}\sinh\frac{\theta}{2}i + \frac{d}{2}\cosh\frac{\theta}{2}j\right).$$

There are many other possible solutions if g_0 is any transformation which commutes with g, that is any other screw motion with the same axis as g, then since $gg_0 = g_0 g$,

$$g = l_1' l_2'$$

where $l_1' = g_0 l_1 \bar{g}_0$ and $l_2' = g_0 l_2 \bar{g}_0$. Notice that the axes of the factors, l_1 and l_2 are perpendicular to the axis of the original screw transformation. Such that,

$$\langle i, k \rangle = 0$$

and

$$< (-\cos h\frac{\theta}{2} i \sin h\frac{\theta}{2} j) + \varepsilon(-\frac{d}{2} \sin h\frac{\theta}{2} i + \frac{d}{2} \cos h\frac{\theta}{2} j), k >= 0$$

The angle between the lines is half the rotation angle of the transformation and the perpendicular distance between the lines is half the translation along the axis of the screw (Selig & Husty, 2011).

5. Line-Symmetric Motions in Lorentzian Space

In Lorentz space, line symmetric motions are defined as follows: take a ruled surface $l(\mu)$ and a fixed coordinate frame, now a line symmetric motion is given by reflecting the fixed frame in consecutive generating lines of the ruled surface, to give a 1-parameter family of frames.

This can be seen as a curve by choosing a starting line in the ruled surface and $l_0 = l(0)$. Now the rigid motion from the frame given by this line to any subsequent line will be

$$g(\mu) l_0 = l(\mu)$$

$$g(\mu) = l(\mu) l_0^{-1} = -l(\mu) l_0$$

since half-turns are self-inverse where $l_0^{-1} = \frac{\bar{l}_0}{|l_0|^2} = -l_0$.

It can be seen that such a curve will satisfy the relation:

$$g(\mu) l_0^{-1} = -l(\mu)$$

$$g(\mu) l_0^{-1} + l(\mu) = 0$$

$$g(\mu) l_0^{-1} + g(\mu) l_0 = 0$$

and

$$g(\mu) l_0^{-1} + \left(\overline{g(\mu) l_0^{-1}}\right) = 0$$

$$g(\mu) l_0^{-1} + \left(\overline{l_0^{-1}}\right) \bar{g}(\mu) = 0$$

$$g(\mu) l_0^{-1} + \left(\overline{-l_0}\right) \bar{g}(\mu) = 0$$

$$g(\mu)l_0^{-1} + l_0\bar{g}(\mu) = 0$$

since any line satisfies $\bar{l} = -l$ and l^2 is a dual number. On the other hand, suppose that $g(\mu)$ is a curve which satisfies the above equation for some line l_0, then

$$g(\mu)l_0 = l_0\bar{g}(\mu)$$

and hence

$$\left(\overline{g(\mu)l_0}\right) = -l_0\bar{g}(\mu) = -(g(\mu)l_0).$$

So $(g(\mu)l_0)$ is a line for every μ and the motion is line symmetric.

A motion which doesn't pass through the identity might still be line symmetric, the motion can always be translated to a path through the identity, that is the motion may have the form $g(\mu) = l(\mu)l_0g_0$ where g_0 is some fixed group element. Such a path will clearly satisfy the equation,

$$g(\mu)\bar{\gamma}_0 + \gamma_0\bar{g}(\mu) = 0$$

where $\gamma_0 = l_0g_0$.

The line symmetric motions can be characterized in another way. If the motion $g(\mu)$ can be factored into a product of two half-turns one of which is fixed then the screw axis of $g(\mu)$ will meet the fixed line at right-angles. Hence the axes of all finite displacements constituting the motion will lie in the congruence of lines meeting and perpendicular to a fixed line. In fact it can be seen that the ruled surface formed by the screw axes of the motion will form a right conoid.

These two characterizations are, of course, equivalent. To see this first recall that two lines l_1 and l_2 will intersect and be perpendicular if and only if they satisfy $l_1\bar{l}_2 + l_2\bar{l}_1 = 0$ (Selig & Husty, 2011).

Now, if the axis of the motion l is intersecting and perpendicular to a line l_0 then clearly $g(\mu)l_0^{-1} + l_0\bar{g}(\mu) = 0$ is satisfied. On the other hand if $g(\mu)l_0^{-1} + l_0\bar{g}(\mu) = 0$ is satisfied then either the lines are intersecting and perpendicular or $\theta = 0$, that is the motion is a pure translation.

This second condition leads to a small but useful result, that a motion about a fixed axis is always line symmetric. This is easily seen since any line coincident and perpendicular to the fixed axis of the motion can be taken as l_0.

For example, now, let give a motion is given as an example of a line symmetric motion. Here this will be verified using the methods developed above. Writing this motion, as a dual quaternion we have,

$$g(\theta) = \left(\cosh\frac{\theta}{2} + k\sinh\frac{\theta}{2}\right) + \varepsilon\left(\sinh\frac{\theta}{2} + k\cosh\frac{\theta}{2}\right)$$

The axis of this motion is always the z-axis. This must be a line symmetric motion. Also we can see that any line perpendicular to the z -axis, for example,

$$l_0 = i$$

this gives a parameterization of the ruled surface as,

$$l(\theta) \;=\; g(\theta)l_0 = [(cosh\frac{\theta}{2} + ksin\,h\frac{\theta}{2}) + \varepsilon(sin\,h\frac{\theta}{2} + kcos\,h\frac{\theta}{2})]i$$

$$= \;(cosh\frac{\theta}{2}i + jsin\,h\frac{\theta}{2}) + \varepsilon(sin\,h\frac{\theta}{2}i + jcos\,h\frac{\theta}{2})$$

If $\theta = 0$ then $l(0) = g(0)l_0 = (\cosh 0 i + j\sinh 0) + \varepsilon(\sinh 0 i + j\cosh 0) = i + \varepsilon j$ (Selig & Husty, 2011; Akyar, 2008; Atasoy et al., 2017).

In conclusion, the exploration of line-symmetric motions within Lorentzian space through the lens of quaternion algebra offers significant insights and advancements in understanding the complex behaviors and properties of objects under relativistic motion. By leveraging the mathematical robustness and computational efficiency of quaternions, this analysis has illuminated the nuanced ways in which objects exhibit symmetry and undergo transformations within the context of special relativity.

Quaternion algebra, with its capacity to encode rotations and Lorentz boosts in a compact and intuitive manner, provides a powerful tool for dissecting and reinterpreting the geometric and algebraic properties of Lorentzian space. This study has demonstrated that quaternions not only simplify the mathematical treatment of line-symmetric motions but also enhance our conceptual grasp of such phenomena, making intricate relativistic effects more accessible and understandable.

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