Chapter 2

Application of q-Differential Transform Method on q-Dirac System[1]

Semanur Karatepe[2]

Fatma Hıra[3]

Abstract

The differential transform method (DTM) is beneficial and practical to obtain exact or approximate solutions to problems involving many different equations, such as linear or non-linear ordinary or partial differential equations, systems of equations, and integral equations. Analysis based on defining the derivative only by the rate of change of functions without using limits is called q-calculus, and examining the concepts in classical analysis in q-calculus is called q-analog. In the literature, the q-analog of the DTM has been examined as the q-DTM. This study presents the application of the q-DTM to the q-Dirac system and the approximate obtaining of the first and second eigenvalues for different q values ($0<q<1$). It is also shown that the results obtained coincide with those corresponding to the classical version of the problem examined for $q \to 1^-$.

1. Introduction

When a new derivative is defined based only on the difference ratio of the functions without using the limit in the classical derivative definition and a new theory is created on this derivative, this opens the doors of q-calculus. For this reason, q-calculus is also known as calculus without limits.

The equivalents of classical expressions such as definition, theorem, and property in q-calculus are called q-analogs (or q-similars). Many existing

1 This chapter is a part of S. Karatepe's Master's thesis.

2 Master Student, Ondokuz Mayıs University, Institute of Graduate Studies, semamattt55@gmail.com, ORCID ID:0009-0000-7148-4768

3 Assoc. Prof., Ondokuz Mayıs University, fatma.hira@omu.edu.tr, ORCID ID: 0000-0003-0724-0704

https://doi.org/10.58830/ozgur.pub430.c1866

classical topics, problems, etc., have been studied in a q-analog manner and continue to be done.

The differential transform method (DTM) is beneficial and preferred, especially in solving nonlinear differential equations. The description and applications of the method can be seen in [6, 20, 23]. The q-analog of this method (as q-DTM) has been examined in one-dimensional q-DTM in [18], two-dimensional q-DTM in [8, 9], and the reduced q-DTM, which is offered as an alternative to two-dimensional DTM, in [17, 21]. The nonlinear damped q-difference equation [19], linear q-deformed Lane-Emdan equation [22], q-Riccati equation [1], q-kinetic equation [10] and the q-Schrödinger equation [11] were solved by these q-DTMs.

Chen and Ho [7] and Hassan [12] used the DTM to solve the eigenvalue problems. Hassan [13] also applied the DTM to the Bratu problem as a nonlinear eigenvalue problem.

The q-Dirac system was introduced in [2], and some of its features were examined in [14, 15]. In this paper, we will apply the q-DTM in the q-Dirac system as the q-eigenvalue problem. As a result, we will obtain the eigenvalues and determine the first and second eigenvalues of the problem for some q-values in the range of $0<q<1$.

2. Preliminaries and Notations

Let $q\in(0,1)$. The q-difference operator D_q [16] is defined by

$$D_q g(x) := \frac{d_q g(x)}{d_q x} = \frac{g(qx)-g(x)}{qx-x} \quad . \tag{2.1}$$

The q-integration is defined by

$$\int_a^b g(x)\,d_q x := \int_0^b g(x)\,d_q x - \int_0^a g(x)\,d_q x, \tag{2.2}$$

where

$$\int_0^x g(t)\,d_q t = \sum_{n=0}^{\infty} x(1-q)q^n g(xq^n), \tag{2.3}$$

provided that the series converges.

The q-trigonometric functions are defined by

$$\cos_q(x) := \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2} x^{2m}}{[2m]_q!} , \tag{2.4}$$

and

$$\sin_q(x) := \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)} x^{2m+1}}{[2m+1]_q!} , \tag{2.5}$$

where $[m]_q = \dfrac{1-q^m}{1-q}$, $[m]_q! = [1]_q [2]_q \ldots [m]_q$ and $[0]_q! = 1$ (see detailed in [3-5]).

The one-dimensional q-DTM of function $y(x)$ is defined as follows [18]

$$Y_q(k) := \frac{1}{[k]_q!} \left(\frac{d_q^k}{d_q x^k} y(x) \right)_{x=0} . \tag{2.6}$$

Here $y(x)$ is called the original function and $Y_q(k)$ is the transformed function. The q-differential inverse transform of $Y_q(k)$ is defined by

$$y(x) = \sum_{k=0}^{\infty} Y_q(k) x^k . \tag{2.7}$$

Let $Y_q(k)$ and $V_q(k)$ be the transformed function of $y(x)$ and $v(x)$, respectively. Then, some basic properties provided by q-DT using the linearity property of q-derivative are given in Table 1.

Tablo 1. Some operations of one-dimensional q-DTM

Function	q- differential transform
$c_1 y(x) \pm c_2 v(x),\ \ c_1, c_2 \in \mathbb{R},$	$c_1 Y_q(k) \pm c_2 V_q(k)$
$x^n,\ n \in \mathbb{N},$	$\delta(k-n),\ \ \delta(k-n) = \begin{cases} 1, & k = n, \\ 0, & k \neq n, \end{cases}$
$\dfrac{d_q y(x)}{d_q x}$	$[k+1]_q Y_q(k+1)$
$\dfrac{d_q^n y(x)}{d_q x^n}$	$[k+1]_q [k+2]_q \ldots [k+n]_q Y_q(k+n)$
$y(qx)$	$q^k Y_q(k)$
$y(x)v(x)$	$\sum_{r=0}^{k} Y_q(r) V_q(k-r)$

3. q-Dirac System and Its Solution with q-DTM

We consider the following q-Dirac system consisting of

$$-\frac{1}{q} D_{q^{-1}} y_2(x) + r(x) y_1(x) = \lambda y_1(x), \tag{3.1}$$

$$D_q y_1(x) + p(x) y_2(x) = \lambda y_2(x), \tag{3.2}$$

with the boundary conditions

$$\gamma_1 y_1(0) + \gamma_2 y_2(0) = 0, \tag{3.3}$$

$$\varphi_1 y_1(a) + \varphi_2 y_2(aq^{-1}) = 0, \tag{3.4}$$

where λ is a complex eigenparameter, γ_i and φ_i $(i=1,2)$ are real numbers, $y(x)=\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $r(x)$ and $p(x)$ are real-valued functions defined on $[0,\ a]$ and continuous at zero (see [2, 14]).

In the following, we consider the special case of the q-Dirac system (3.1)-(3.4) in which $r(x)=0$ and $p(x)=0$,

$$-\frac{1}{q}D_{q^{-1}}y_2(x)=\lambda y_1(x), \tag{3.5}$$

$$D_q y_1(x)=\lambda y_2(x), \tag{3.6}$$

with

$$y_1(0)-y_2(0)=0, \tag{3.7}$$

$$y_2(\pi q^{-1})=0. \tag{3.8}$$

If the relation $D_{q^{-1}}y(x)=(D_q y)(xq^{-1})$ is used in (3.5) and x replaced by qx, then (3.5) can be written as

$$-\frac{1}{q}D_q y_2(x)=\lambda y_1(qx). \tag{3.9}$$

Taking q-DT of (3.6) and (3.9), we get

$$[k+1]_q\, Y_{1,q}(k+1)=\lambda Y_{2,q}(k), \tag{3.10}$$

$$-\frac{1}{q}[k+1]_q\, Y_{2,q}(k+1)=\lambda q^k Y_{1,q}(k), \tag{3.11}$$

where $Y_{1,q}(k)$ and $Y_{2,q}(k)$ are transformed functions of $y_1(x)$ and $y_2(x)$, respectively. (3.10) and (3.11) gives the recurrence relation

$$\begin{cases} Y_{1,q}\left(k+1\right)=\dfrac{\lambda}{\left[k+1\right]_q}Y_{2,q}\left(k\right), \\ Y_{2,q}\left(k+1\right)=-\dfrac{\lambda q^{k+1}}{\left[k+1\right]_q}Y_{1,q}\left(k\right). \end{cases} \tag{3.12}$$

Applying q -DTM on (3.7) and put

$$Y_{1,q}\left(0\right)=c, \tag{3.13}$$

then it becomes

$$Y_{2,q}\left(0\right)=c. \tag{3.14}$$

For $k=0$, using (3.13) and (3.14) into (3.12), we get

$$\begin{aligned} Y_{1,q}\left(1\right)&=\frac{\lambda c}{\left[1\right]_q}, \\ Y_{2,q}\left(1\right)&=-\frac{\lambda qc}{\left[1\right]_q}. \end{aligned} \tag{3.15}$$

Similarly, for $k=1$, using (3.15) into (3.12), we get

$$\begin{aligned} Y_{1,q}\left(2\right)&=-\frac{\lambda^2 qc}{\left[2\right]_q!}, \\ Y_{2,q}\left(2\right)&=-\frac{\lambda^2 q^2 c}{\left[2\right]_q!}, \end{aligned} \tag{3.16}$$

and for $k=2$, using (3.16) into (3.12), we have

$$\begin{aligned} Y_{1,q}\left(3\right)&=-\frac{\lambda^3 q^2 c}{\left[3\right]_q!}, \\ Y_{2,q}\left(3\right)&=\frac{\lambda^3 q^4 c}{\left[3\right]_q!}. \end{aligned} \tag{3.17}$$

Following the same procedure we obtain for $k=0,1,2,\ldots$

$$\begin{cases} Y_{1,q}(2k) = \dfrac{(-1)^k \lambda^{2k} q^{k^2} c}{[2k]_q!}, \\ Y_{2,q}(2k) = \dfrac{(-1)^k \lambda^{2k} q^{k(k+1)} c}{[2k]_q!}, \end{cases} \tag{3.18}$$

$$\begin{cases} Y_{1,q}(2k+1) = \dfrac{(-1)^k \lambda^{2k+1} q^{k(k+1)} c}{[2k+1]_q!}, \\ Y_{2,q}(2k+1) = \dfrac{(-1)^{k+1} \lambda^{2k+1} q^{(k+1)^2} c}{[2k+1]_q!}. \end{cases} \tag{3.19}$$

Hence, using (2.7), we have

$$\begin{aligned} y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} &= \begin{pmatrix} \sum\limits_{k=0}^{\infty} Y_{1,q}(k) x^k \\ \sum\limits_{k=0}^{\infty} Y_{2,q}(k) x^k \end{pmatrix} \\ &= \begin{pmatrix} \sum\limits_{k=0}^{\infty} \left(\dfrac{(-1)^k \lambda^{2k} q^{k^2} c}{[2k]_q!} x^{2k} + \dfrac{(-1)^k \lambda^{2k+1} q^{k(k+1)} c}{[2k+1]_q!} x^{2k+1} \right) \\ \sum\limits_{k=0}^{\infty} \left(\dfrac{(-1)^k \lambda^{2k} q^{k(k+1)} c}{[2k]_q!} x^{2k} + \dfrac{(-1)^{k+1} \lambda^{2k+1} q^{(k+1)^2} c}{[2k+1]_q!} x^{2k+1} \right) \end{pmatrix}. \end{aligned} \tag{3.20}$$

In order for (3.20) to be the solution of the q -Dirac system (3.5)-(3.8), the condition (3.8) must be satisfied. Additionally, if $c \neq 0$ is accepted since the eigenfunctions will be non-trivial solutions, and using (2.4) and (2.5), we get

$$\cos_q\left(\lambda q^{-1/2}\pi\right) - q^{1/2} \sin_q\left(\lambda q^{-1/2}\pi\right) = 0\,. \tag{3.21}$$

Thus, the eigenvalues of problem (3.5)-(3.8) are the roots of equation (3.21). However, we cannot clearly determine the roots of this equation as in the classical way. The first two eigenvalues will be approximately calculated with the help of approximate solution according to the method stated below.

To approximately calculate the first and second eigenvalues of problem (3.5)-(3.8) or the first and second roots of equation (3.21), the following procedure is followed. If we calculate the m th term for $Y_{2,q}(k)$ using (3.20), and substitute them in (3.8), we obtain

$$\sum_{k=0}^{m} Y_{2,q}(k)\pi^k q^{-k} = 0, \tag{3.22}$$

Hence, we can write

$$cf_q^{(m)}(\lambda) = 0, \tag{3.23}$$

where $f_q^{(m)}(\lambda)$ is a polynomial of λ corresponding to m. Let $c \neq 0$, then we have

$$f_q^{(m)}(\lambda) = 0. \tag{3.24}$$

Solving (3.24), we have $\lambda = \lambda_l^{(m)}$, $l = 1,2,3,\ldots$, where $\lambda_l^{(m)}$ is the l th estimated eigenvalue corresponding to m, and m is indicated by

$$\left|\lambda_l^{(m)} - \lambda_l^{(m-1)}\right| \leq \delta, \tag{3.25}$$

where $\lambda_l^{(m-1)}$ is the l th estimated eigenvalue corresponding to $m-1$ and δ is a small value. If (3.25) is not satisfied, then substituting $(m+1)$ for m and following the same procedure as shown in (3.23)-(3.25).

Case 1: For $q = 0.9$.

From (3.12) and (3.22)-(3.24), if the approximate solution is calculated to $m = 6$, then we have

$$f_q^{(6)}(\lambda) = 1 - \lambda\pi - \frac{\lambda^2\pi^2}{[2]_q!} + \frac{\lambda^3\pi^3 q}{[3]_q!} + \frac{\lambda^4\pi^4 q^2}{[4]_q!} - \frac{\lambda^5\pi^5 q^4}{[5]_q!} - \frac{\lambda^6\pi^6 q^6}{[6]_q!} = 0. \tag{3.26}$$

Solving (3.26) for $q = 0.9$, and take the real root

$$\lambda_1^{(6)} = 0.2475. \tag{3.27}$$

By the same way, when $m = 5$, the root of (3.26) becomes

$$\lambda_1^{(5)} = 0.2476. \tag{3.28}$$

From (3.27) and (3.28), we obtain

$$\left|\lambda_1^{(6)} - \lambda_1^{(5)}\right| = 0.0001 \leq \delta, \tag{3.29}$$

where δ is a sufficiently small value that we determine. From (3.29), we get $\lambda_1 = 0.2475$ is the first eigenvalue for $q = 0.9$. Substituting λ_1 into (3.20) for $m = 6$ we obtain the first eigenfunction.

Following the procedure mentioned above, if we solve the resulting equation for $m = 12$

$$f_q^{(12)}(\lambda) = 1 - \lambda\pi - \frac{\lambda^2\pi^2}{[2]_q!} + \frac{\lambda^3\pi^3 q}{[3]_q!} + \frac{\lambda^4\pi^4 q^2}{[4]_q!} - \frac{\lambda^5\pi^5 q^4}{[5]_q!} - \frac{\lambda^6\pi^6 q^6}{[6]_q!} + \frac{\lambda^7\pi^7 q^9}{[7]_q!} + \frac{\lambda^8\pi^8 q^{12}}{[8]_q!} - \frac{\lambda^9\pi^9 q^{16}}{[9]_q!} - \frac{\lambda^{10}\pi^{10} q^{20}}{[10]_q!} + \frac{\lambda^{11}\pi^{11} q^{25}}{[11]_q!} + \frac{\lambda^{12}\pi^{12} q^{30}}{[12]_q!} = 0 \tag{3.30}$$

and take the real root, we have

$$\lambda_1^{(12)} = 0.2475, \tag{3.31}$$

$$\lambda_2^{(12)} = 1.2172. \tag{3.32}$$

Note that $\lambda_1^{(12)} = \lambda_1^{(6)}$. Due to $\left|\lambda_2^{(12)} - \lambda_2^{(11)}\right| = 0.004 \le \varepsilon$, we have the second eigenvalue $\lambda_2 = 1.2172$ for $q = 0.9$. Substituting λ_2 into (3.20) for $m = 12$ we obtain the second eigenfunction.

Case 2: For $q = 0.8$ and $q = 1$.

The equations (3.26) and (3.30) solving for $q = 0.8$ and $q = 1$, respectively, and following the same procedure as Case 1, the first and second eigenvalues given in Table 2 are obtained.

Tablo 2. The first and second eigenvalues for $q = 0.8$ and $q = 1$.

Eigenvalues	q- values ($0 < q < 1$)	
	$q = 0.8$	$q = 1$
λ_1	0.2447	$0.2499 \cong 0.25$
λ_2	1.1779	$1.2477 \cong 1.25$

For the classical form of the q -Dirac system (3.5)-(3.8), when $q \to 1^-$, which is

$$\begin{cases} -y_2'(x) = \lambda y_1(x), \\ y_1'(x) = \lambda y_2(x), \end{cases}$$

$$y_1(0) - y_2(0) = 0,$$

$$y_2(\pi) = 0 \cdot$$

If this system is solved analytically, its eigenvalues are the roots of the equation $\tan(\lambda\pi) = 1$ and they have the form $\lambda_n = \frac{1}{4}(1+4n),\ n \in \mathbb{Z}$. Thus, the first and second eigenvalues are 0.25 and 1.25, respectively, which are the same for $q = 1$ in Tablo 2.

Similar investigations can be made for different q values in $0 < q < 1$, and for more m steps.

REFERENCES

1. Abdulmajeed, A.Y. and Khudair, A.R. Solving Riccati type q-difference equations via difference transform method. Al-Qadisiyah Journal of Pure Science, (2021), 26:4.
2. Allahverdiev, B.P. and Tuna, H. One-dimensional q-Dirac equation, Math. Meth. Appl. Sci., 40 (2017), 7287–7306.
3. Andrews, G.E., Askey, R. and Roy, R. Special Functions, Cambridge Univ. Press, Cambridge, 1999.
4. Annaby, M. H. and Mansour, Z.S. On the zeros of basic finite Hankel transforms, J. Math. Anal. Appl., 323 (2006), 1091-1103.
5. Annaby, M. H. and Mansour, Z. S. A basic analog of a theorem of P´olya, Math. Z., 258 (2008), 363-379.
6. Arikoglu, A. and Ozkol, I. Solution of boundary value problems for integro-differential equations by using differential transform method, Appl. Math. Comput., 168 (2), (2005), 1145-1158.
7. Chen, C. K and Ho, S. H. Application of differential transformation to eigenvalue problems, Appl. Math. Comput., 79 (1996), 173-188.
8. El-Shahed, M. and Gaber, M. Two-dimensional -differential transformation and its application, Appl. Math. Comput., 217, (2011), 9165-9172.
9. Garg, M. and Chanchlani, L. On two-dimensional q-differential transform method, Afr. Mat., 25 (2014), 529-538.
10. Garg, M. and Chanchlani, L. On fractional q-kinetic equation, Mat. Bilt., 36 (2012), 33-46.
11. Garg, M. and Chanchlani, L. Solution of nonlinear q-Schrodinger equation by two-dimensional q-differential transform method, Kuwait J. Sci., 40:2, (2013), 17-30.
12. Hassan, I. On solving some eigenvalve problems by using differential transformaion. Appl. Math. Comput., 127(1), (2002), 22.
13. Hassan, I. Applying differential transformation method to the one-dimensional planar Bratu problem. Inter. J. Contemp. Math. Sci. 2(30), (2007), 1493-1504.
14. Hıra, F. Eigenvalues and eigenfunctions of q-Dirac system, J. Sci. Arts 4 (45) (2018), 963-972.
15. Hıra, F. Sampling theorem associated with q-Dirac system, J. Math. Comput. Sci. 9 (2), (2019), 182-193
16. Jackson, F. H. On q-definite integrals, The Quarterly Journal of Pure and Applied Mathematics. 41, (1910), 193-203.
17. Jafari, H., Haghbin, A., Hesam, S. and Baleanu, D. Solving partial q-differential equations within reduced q-differential transformation method, Rom. Journ. Phys., 59:5-6, . (2014), 399-407.

18. Jing, S. C. and Fan, H. Y. q-Taylor's formula with its q-remainder, Commun. Theor. Phys., 23, (1995), 117-120.
19. Liu, H. K. Application of a differential transformation method to strongly nonlinear damped q-difference equations, Comput. Math. Appl., 61, (2011), 2555-2561.
20. Qin, Y., and Lou, Q. Differential transform method for the solutions to some initial value problems in chemistry, J. Math. Chemist., 59, (2021), 1046-1053.
21. Sadik, M. O. and Orie, B. O. Application of q-calculus to the solution of partial q-differential equations, Applied Mathematics, 12, (2004), 669-678.
22. Yener, G. and Emiroğlu, I. q-deformed Lane Emden differential equations and its solution by q-differential transform method, Int. J. Contemp. Math. Sci., 9:4, (2014), 163-174.
23. Zhou, J. K. Differential transformation method and its application for electrical circuits, Hauzhang University Press, Wuhan, China (1), (1986), 96-102.