Bölüm 1

Homothetic Motions With Dual Bicomplex Numbers in Dual 4-Space

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Abstract

In this paper, we define a dual matrix for dual bicomplex numbers in dual 4-space. By using this new dual matrix, we define a new dual motion and it is proven that this dual motion is homothetic. We provide some theorems for dual velocities, dual pole points, and dual pole curves for this one parameter dual homothetic motion. Moreover, we demonstrate that the motion described by the regular order n dual curve has only one acceleration center of order (n-1) at every t-instant after defining dual accelerations.

1. Introduction

In 1873, Clifford defined dual numbers and were used at the beginning of the twentieth century by the German mathematician Eduard Study, who used them to represent the dual angle which measures the relative position of two skew lines in space (Guggenheimer, 1963; Kotelnikov, 1895; Study, 1901). In the following years, dual numbers are used in the investigation of instantaneous screw axes with the help of dual transformations. The Italian mathematician Gerolamo Cardano first gave complex numbers while trying to solve a simpler state of the cubic equation. After, Leonard Euler illustrated the complex numbers as points with rectangular coordinates by using the notation $i_1^2 = -1$. Bicomplex numbers were defined in 1892 to improve the properties of algebra. As a result of the research, it was included in an article by Corrado Segre. Here bicomplex numbers are considered as tricomplex

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numbers. Rochon and Tremblay presented a study titled "II. The Hilbert Space," which was based on bicomplex quantum mechanics and then, Rochon and Shapiro gave algebraic properties of bicomplex and hyperbolic numbers (Rochon et al. 2004, 2006; Price,1991). This study presents bicomplex (hyperbolic) numbers from several perspectives on Hilbert Space in quantum physics. Any set of bicomplex numbers can be given by

$$\mathcal{C}_2 = \{A = a \ + b\ i_1 + c\ i_2 + \ di_3\}$$

where $a\ , b\ , c\ , d \in \mathbb{R}$, the imaginary units $i_1,\ i_2$ and i_3 are governed by the rules:

$$i_1^2 = i_2^2 = -1, \qquad i_1 i_2 = i_2 i_1 = 1.$$

A dual bicomplex number is defined as a dual complex number depending on four units

$$\tilde{A} = \ \tilde{a}_0 + \tilde{a}_1 i_1 + \tilde{a}_2 i_2 + \tilde{a}_3 i_3$$

$$= (a_0 + a_0^*\ \varepsilon) + (a_1 + a_1^*\ \varepsilon) i_1 + (a_2 + a_2^*\ \varepsilon) i_2 + (a_3 + a_3^*\ \varepsilon) i_3$$

where i_1, i_2, i_3 are the imaginary units and ε is the dual unit which satisfy the conditions

$$i_1^2 = i_2^2 = -1,\ i_1 i_2 = i_2 i_1 = i_3 \text{ and } \varepsilon^2 = 0.$$

The set of all dual bicomplex number is defined by

$$\mathbb{C}_2^{\mathbb{D}} = \left\{\tilde{A} \mid \tilde{A} = \tilde{a}_0 + \tilde{a}_1 i_1 + \tilde{a}_2 i_2 + \tilde{a}_3 i_3 \colon \tilde{a}_{0-3} \in \mathbb{D}\right\}.$$

Proposition 1. Let $\tilde{A}$ and $\tilde{B}$ be dual bicomplex numbers, then their addition and multiplication are

$$\tilde{A}+\ \tilde{B}=\left(\tilde{a}_0+\tilde{b}_0\right)+\left(\tilde{a}_1+\tilde{b}_1\right)i_1+\left(\tilde{a}_2+\tilde{b}_2\right)i_2+\left(\tilde{a}_3+\tilde{b}_3\right)i_3$$

and

$$\tilde{A}\tilde{B}=(\tilde{a}_0+\tilde{a}_1i_1+\tilde{a}_2i_2+\tilde{a}_3i_3)\left(\tilde{b}_0+\tilde{b}_1i_1+i_2\tilde{b}_2+i_3\tilde{b}_3\right)$$

$$=\left(\tilde{a}_0\tilde{b}_0-\tilde{a}_1\tilde{b}_1-\tilde{a}_2\tilde{b}_2+\tilde{a}_3\tilde{b}_3\right)+\left(\tilde{a}_0\tilde{b}_1+\tilde{a}_1\tilde{b}_0-\tilde{a}_2\tilde{b}_3-\tilde{a}_3\tilde{b}_2\right)i_1$$

$$+\left(\tilde{a}_0\tilde{b}_2-\tilde{a}_2\tilde{b}_0+\tilde{a}_3\tilde{b}_1-\tilde{a}_1\tilde{b}_3\right)i_2$$
$$+\left(\tilde{a}_0\tilde{b}_3+\tilde{a}_3\tilde{b}_0+\tilde{a}_1\tilde{b}_2+\tilde{a}_2\tilde{b}_1\right)i_3$$

According to the imaginary unitsi_1, i_2 and i_3, the conjugates and norms of the dual bicomplex number $\tilde{A}$, are

$$\tilde{A}^{\boldsymbol{i}_1}=\tilde{a}_0-\tilde{a}_1i_1+\tilde{a}_3i_2-\tilde{a}_3i_3$$

$$\widetilde{A}\,\tilde{A}^{\boldsymbol{i}_1}=\tilde{a}_0^2+\tilde{a}_1^2-\tilde{a}_2^2-\tilde{a}_3^2+2i_2(\tilde{a}_0\tilde{a}_2+\tilde{a}_1\tilde{a}_3)$$

$$\tilde{A}^{\boldsymbol{i}_2}=\tilde{a}_0+\tilde{a}_1i_1-\tilde{a}_3i_2-\tilde{a}_3i_3$$

$$\widetilde{A}\,\tilde{A}^{\boldsymbol{i}_2}=\tilde{a}_0^2-\tilde{a}_1^2+\tilde{a}_2^2-\tilde{a}_3^2+2i_2(\tilde{a}_0\tilde{a}_1+\tilde{a}_2\tilde{a}_3)$$

$$\tilde{A}^{\boldsymbol{i}_3}=\tilde{a}_0-\tilde{a}_1i_1-\tilde{a}_3i_2+\tilde{a}_3i_3$$

$$\widetilde{A}\,\tilde{A}^{\boldsymbol{i}_3}=\tilde{a}_0^2+\tilde{a}_1^2+\tilde{a}_2^2+\tilde{a}_3^2+2i_2(\tilde{a}_0\tilde{a}_3-\tilde{a}_1\tilde{a}_2)$$

and

$$\left\|\tilde{A}\right\|_{i_1} = \tilde{a}_0^2 + \tilde{a}_1^2 - \tilde{a}_2^2 - \tilde{a}_3^2$$

$$\left\|\tilde{A}\right\|_{i_2} = \tilde{a}_0^2 - \tilde{a}_1^2 + \tilde{a}_2^2 - \tilde{a}_3^2$$

$$\left\|\tilde{A}\right\|_{i_3} = \tilde{a}_0^2 + \tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_3^2$$

The system $\mathbb{C}_2^{\mathbb{D}}$ is a commutative algebra. It is referred as the dual bicomplex number algebra shown with $\mathbb{C}_2^{\mathbb{D}}$, briefly one of the bases of this algebra is $\{1, i_1, i_2, i_3\}$ and the dimension is 4. It is possible to give the production similar to Hamilton operators which has defined (Agrawal, 1987; Hacısalihoglu, 1980,1983; Yayli Y,1995).

$$\phi: \tilde{A} = \tilde{a}_0 + \tilde{a}_1 i_1 + \tilde{a}_2 i_2 + \tilde{a}_3 i_3 \in \mathbb{C}_2^{\mathbb{D}} \to \phi(\tilde{A}) = \begin{bmatrix} \tilde{a}_0 & -\tilde{a}_1 & -\tilde{a}_2 & \tilde{a}_3 \\ \tilde{a}_1 & \tilde{a}_0 & -\tilde{a}_3 & -\tilde{a}_2 \\ \tilde{a}_2 & -\tilde{a}_3 & \tilde{a}_0 & -\tilde{a}_1 \\ \tilde{a}_3 & \tilde{a}_2 & \tilde{a}_1 & \tilde{a}_0 \end{bmatrix}$$

$\mathbb{C}_2^{\mathbb{D}}$ is algebraically isomorphic to the matrix algebra

$$\mathcal{R} = \phi(\tilde{A}) = \left\{ \begin{bmatrix} \tilde{a}_0 & -\tilde{a}_1 & -\tilde{a}_2 & \tilde{a}_3 \\ \tilde{a}_1 & \tilde{a}_0 & -\tilde{a}_3 & -\tilde{a}_2 \\ \tilde{a}_2 & -\tilde{a}_3 & \tilde{a}_0 & -\tilde{a}_1 \\ \tilde{a}_3 & \tilde{a}_2 & \tilde{a}_1 & \tilde{a}_0 \end{bmatrix} : \ \tilde{a}_0, \tilde{a}_1, \ \tilde{a}_2, \ \tilde{a}_3 \in \mathbb{D} \right\}$$

and $\phi(\tilde{A})$ is a faithful real matrix representation of $\mathcal{R}$. In this paper, we give a dual matrix which is similar to Hamilton operators for dual bicomplex numbers in dual 4-space. Thanks to this new dual matrix, we define a new dual motion and it is proven that this dual motion is homothetic. We provide some theorems for dual velocities, dual pole points, and dual pole curves for this one parameter dual homothetic motion. Moreover, we demonstrate that the motion described by the regular order n dual curve has only one acceleration center of order $(n-1)$ at every t-instant after defining dual accelerations

2. Homothetic Motions in dual 4-space

Definition 1. Let $\mathcal{M}$ and $\mathcal{S}^3$ be a dual hypersurface and unit dual sphere, respectively, as following,

$$\mathcal{M} = \{\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3) \mid \gamma_0\gamma_3 - \gamma_1\gamma_2 = 0\},$$

$$\mathcal{S}^3 = \{\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3) \mid \gamma_0^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1\}$$

From definition (1), let us consider the following dual curve:

If $\gamma \quad : I \subset R \rightarrow \mathcal{M} \subset \mathbb{D}^4$ given by

$$t \rightarrow \gamma(t) = \gamma_0(t) + \gamma_1(t)i_1 + \gamma_2(t)i_2 + \gamma_3(t)i_3$$

$$= \left(a_0(t) + \varepsilon a_0^*(t)\right) + \left(a_1(t) + \varepsilon a_1^*(t)\right)i_1 + \left(a_2(t) + \varepsilon a_2^*(t)\right)i_2 + \left(a_3(t) + \varepsilon a_3^*(t)\right)i_3$$

$$= (a_0(t) + a_1(t)i_1 + a_2(t)i_2 + a_3(t)i_3) + \varepsilon(a_0^*(t) + a_1(t)^*i_1 + a_2^*(t)i_2 + a_3^*(t)i_3)$$

$$= \overrightarrow{\mathbb{A}} + \varepsilon\overrightarrow{\mathbb{A}}^*$$

for every $t \in I$. We suppose that the curve $\gamma(t)$ is differentiable dual regular curve of order n. The operator $\mathcal{P} = \mathcal{R}^+ = \mathcal{R}^- = \mathcal{R}$, corresponding to $\gamma(t)$, is defined by the following dual matrix:

$$\mathcal{R} = \begin{bmatrix} \gamma_0 & -\gamma_1 & -\gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_0 & -\gamma_3 & -\gamma_2 \\ \gamma_2 & -\gamma_3 & \gamma_0 & -\gamma_1 \\ \gamma_3 & \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix} \tag{1}$$

Let $\|\gamma'(t)\| = 1$, $\gamma(t)$ be a unit velocity dual curve. If $\gamma(t)$ does not pass through the orijin, and $\gamma(t) \neq 0$, from Equality (1), the matrix can be represent as

$$\mathcal{P} = \lambda\, \mathcal{Q}, \tag{2}$$

$$\mathcal{P} = \lambda \begin{bmatrix} \gamma_0/\lambda & -\gamma_1/\lambda & -\gamma_2/\lambda & \gamma_3/\lambda \\ \gamma_1/\lambda & \gamma_0/\lambda & -\gamma_3/\lambda & -\gamma_2/\lambda \\ \gamma_2/\lambda & -\gamma_3/\lambda & \gamma_0/\lambda & -\gamma_1/\lambda \\ \gamma_3/\lambda & \gamma_2/\lambda & \gamma_1/\lambda & \gamma_0/\lambda \end{bmatrix}$$

and

$$\lambda : I \subset \mathbb{R} \to \mathbb{D}$$
$$t \to \lambda(t) = \| \gamma(t) \| = \sqrt{|\gamma_0^2(t) + \gamma_1^2(t) + \gamma_2^2(t) + \gamma_3^2(t)|} \text{ and } \gamma(t) \neq 0.$$

Theorem 1. From Equalities (1) and (2), the matrix $\mathcal{Q}$ is dual orthogonal matrix in $\mathbb{D}^4$.

Proof. Let $\gamma_0(t)\gamma_3(t) - \gamma_1(t)\gamma_2(t) = 0$. In Equality $\mathcal{P} = \lambda\mathcal{Q}$, the matrix $\mathcal{Q}$ has been shown by $\mathcal{Q}^T\, \mathcal{Q} = I_4$ where, the matrix $\mathcal{Q}$ is dual orthogonal matrix and $det\mathcal{Q} = 1$.

3. A dual motion with one parameter in dual 4-space

Let the fixed space and the motinal space be, respectively, $\mathcal{K}_0$ and $\mathcal{K}$. In this case, one-parametric motion of $\mathcal{K}_0$ with respect to $\mathcal{K}$ will be denoted by $\mathcal{K}_0/\mathcal{K}$. This motion can be expressed by

$$\begin{bmatrix} \mathcal{X} \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda\mathcal{Q} & \mathcal{C} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{X}_0 \\ 1 \end{bmatrix} \quad (3)$$

or equivalently

$$\mathcal{X} = \lambda\mathcal{Q}\mathcal{X}_0 + \mathcal{C} \quad (4)$$

where, $\mathcal{X}$ and $\mathcal{X}_0$ represent position vectors of any point, respectively, in $\mathcal{K}$ and $\mathcal{K}_0$, and $\mathcal{C}$ represent any translation vector.

Definition 2. In dual 4-space, the one-parameter dual homothetic motion of a body is generated by the transformation given in Equalities (3) and (4). Here λ is called the homothetic scale, which is a dual scalar matrix, $\mathcal{Q}$ is an 4×4 dual orthogonal matrix, $\mathcal{X}_0$ and $\mathcal{C}$ are 4×1 dual matrices, and $\mathcal{Q}, \mathcal{C}$, and λ are of class $\mathcal{C}^{\mathrm{n}}$. In order not to encounter the case of affine transformation we suppose that

$$\lambda(t) = h(t) + \varepsilon h^*(t) \neq \text{constant}, \ \lambda(t) \neq 0$$

and to prevent the cases of pure rotation and pure translation we also suppose that

$$\dot{\lambda}\mathcal{C} + \lambda\dot{\mathcal{C}} \neq 0, \dot{\mathcal{C}} \neq 0$$

Corollary 1. In dual space $\gamma(t) \in \mathbb{D}^4$, The homothetic motions are regular and has only one instantaneous rotation centre at all time t .

Theorem 3. The motion defined by Equality (3) is a dual homothetic motion with one parameter.

Proof. The matrix determined by the equation in Equality (3) is a dual homothetic motion with one parameter, where $\mathcal{Q} \in \mathrm{SO}(4)$.

Theorem 4. Let $\gamma(t)$ be a unit velocity curve and $\dot{\gamma}(t) \in \mathcal{M}$ then the derivation operator $\dot{\mathcal{P}}$ of $\mathcal{P}$ is dual orthogonal matrix in $\mathbb{D}^4$.

Proof. Since $\gamma(t)$ is a dual unit velocity curve,

$$\dot{\gamma}_0^2(t) + \dot{\gamma}_1^2(t) + \dot{\gamma}_2^2(t) + \dot{\gamma}_3^2(t) = 1$$

and $\dot{\gamma}(t) \in \mathcal{M}$, then

$$\dot{\gamma}_0(t)\dot{\gamma}_3(t) - \dot{\gamma}_1(t)\dot{\gamma}_2(t) = 0.$$

Thus, $\dot{\mathcal{P}}\dot{\mathcal{P}}^T = \dot{\mathcal{P}}^T\dot{\mathcal{P}}$ and $det\dot{\mathcal{P}} = 1$.

Theorem 5. If $\gamma(t)$ is a dual spherical curve on $\mathcal{M}$, then the motion is rotatin motion.

Proof. Since $\gamma(t)$ is a dual spherical curve on $\dot{\gamma}_0^2(t) + \dot{\gamma}_1^2(t) + \dot{\gamma}_2^2(t) + \dot{\gamma}_3^2(t) = 1$ and

$\mathcal{P}\mathcal{P}^T = \mathcal{P}^T\mathcal{P}$, $\mathcal{P}$ is a dual orthogonal matrix and $det\mathcal{P} = 1$. Thus $\mathcal{P}$ is a dual rotating matrix in dual space $\mathbb{D}^4$ also, the value of $\det\dot{\mathcal{P}}$ is independent of λ.

4. Dual velocities, dual pole points and dual pole curves

From Equality (3) we obtain

$$\mathcal{X} = \mathcal{P}\mathcal{X}_0 + \mathcal{C} \tag{5}$$

then

$$\mathcal{X}_0 = -\mathcal{P}^{-1}(\mathcal{X} - \mathcal{C}).$$

If we let $\mathcal{C}' = \mathcal{P}^{-1}\mathcal{C}$, Then $(\lambda^{-1} = \frac{1}{\lambda}\mathrm{I},\ \lambda^{\mathrm{T}} = \lambda)$ cause

$$\mathcal{X}_0 = \mathcal{P}^{-1}\mathcal{X} + \dot{\mathcal{C}} \tag{6}$$

Equalities (5) and (6) are coordinate transformations between the fixed and moving dual spaces. Differentiating Equality (6) with respect to t we get

$$\dot{\mathcal{X}} = \dot{\mathcal{P}}\mathcal{X}_0 + \dot{\mathcal{C}} + \dot{\mathcal{X}}_0 + \mathcal{P}\dot{\mathcal{X}}_0$$

where $\mathcal{P}\dot{\mathcal{X}}_0$ is the dual relative velocity, $\dot{\mathcal{P}}\mathcal{X}_0 + \dot{\mathcal{C}}$ is the dual sliding velocity, and $\dot{\mathcal{X}}$ is the dual absolute velocity of point $\dot{\mathcal{X}}_0$. In this case the following theorem can be given.

Theorem 6. For dual homothetic motion with one parameter in dual 4-space, the dual absolute velocity vector of a moving system of point $\dot{\mathcal{X}}_0$ at that time t is the sum of the dual sliding velocity and dual relative velocity of $\dot{\mathcal{X}}_0$.

To find the pole point, we have to solve the equation

$$\dot{\mathcal{P}}\mathcal{X}_0 + \dot{\mathcal{C}} = 0.$$

Any solution of above equation is a dual pole point of the dual motion at that t - instant, which is the only solution. In that case the following theorem can be given.

Theorem 7. In $\mathcal{K}_0$, if $\gamma(t)$ is a dual unit velocity curve and $\dot{\gamma}(t) \in \mathcal{M}$, then the dual pole point corresponding to each t-instant is the rotation by $\dot{\mathcal{P}}$ of the dual speed vector $\dot{\mathcal{C}}$ of the translation vector at that moment.

Proof. Since the matrix $\dot{\mathcal{P}}$ is dual orthogonal, then the matrix $\dot{\mathcal{P}}^{\mathrm{T}}$ is dual orthogonal. Thus it makes a dual rotation.

5. Dual accelerations and dual acceleration centers

Definition 3. The set of the zeros of dual sliding acceleration of order n is defined the dual acceleration centre of order $(n-1)$. By the above definition, we have to solve the solution of the equation

$$\mathcal{P}^{(n)}\mathcal{X}_0 + \mathcal{C}^{(n)} = \sum_{k=o}^{n} \binom{n}{k} \lambda^{(n-k)} \mathcal{Q}^{(n)} \mathcal{X}_0 + \mathcal{C}^{(n)} = 0 \qquad (9)$$

where $\mathcal{P}^{(n)} = \frac{d^n \mathcal{P}}{dt^n}$ and $\mathcal{C}^{(n)} = \frac{d^n \mathcal{C}}{dt^n}$. We know that $\gamma(t)$ is a regular curve of order n and

$\gamma^{(n)} \in \mathcal{M}$. Then we have $\gamma_0^{(n)}\gamma_3^{(n)} + \gamma_1^{(n)}\gamma_2^{(n)} = 0$. Thus,

$$\left(\gamma_0^{(n)}\right)^2 + \left(\gamma_1^{(n)}\right)^2 + \left(\gamma_2^{(n)}\right)^2 + \left(\gamma_3^{(n)}\right)^2 \neq 0$$

Also, we have

$$det\mathcal{P}^{(n)} = \left(\left(\gamma_0^{(n)}\right)^2 + \left(\gamma_1^{(n)}\right)^2 + \left(\gamma_2^{(n)}\right)^2 + \left(\gamma_3^{(n)}\right)^2\right)^2 \neq 0.$$

Thus matrix $\mathcal{P}^{(\mathrm{n})}$ has an inverse and by Equality (9), the dual acceleration centre of order

$(n-1)$ at every t-instant, is

$$\mathcal{X}_0 = \left[\mathcal{P}^{(n)}\right]^{-1}\left[-\mathcal{P}^{(\mathrm{n})}\right].$$

Example 1. Let $\gamma: I \subset R \rightarrow \mathcal{M} \subset \mathbb{D}^4$ be a dual curve given by

$\gamma(t) = \frac{1}{\sqrt{2}}(cost, sint, cost, sint)$. Note that $\gamma(t) \in \mathcal{S}^3$ and since $\|\dot{\gamma}(t)\| = 1$, then $\gamma(t)$ is a unit velocity curve. Moreover,

$$\dot{\gamma}(t), \ddot{\gamma}(t),,, \gamma^{(n)}(t) \in \mathcal{M}.$$

Thus $\gamma(t)$ satisfies all conditions of the above theorems.

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