Chapter 3

Quantum Calculus Approach to the Dual Number Sequences

Faik Babadağ[1]

Abstract

In recent years, many researchers have focused on the chaotic dynamics of quantum calculus, which arises in a variety of areas including the study of fractals, multi-fractal measures, combinatorics and special functions.

In this paper, owing to some useful q-calculus notations, we consider the sequence of q-Fibonacci dual number and q-Lucas dual number with a different perspective and we present some formulas, facts, and properties about these number sequences. After that, some fundamental identities are given, such as D' ocagnes, Cassini, Catalan, and Binet formulas and relations of the q-dual number sequences, and defined the new dual polynomial and function called q-dual Fibonacci polynomial and function sequences. Then, we provide some properties for these sequences.

1. Introduction

W. K. Clifford (1845-1879) introduced the algebra of dual numbers as a tool for his geometrical investigations, and Kotelnikov gave the first applications (Kotelnikov, 1985). Eduard Study gave line geometry and kinematics using dual numbers and dual vectors (Study, 2022). He showed that the points of the dual unit sphere in $\mathbb{D}^3$ have a one-to-one relationship. Dual numbers have modern applications in computer modeling of rigid bodies, mechanism design, kinematics, human body modeling, and dynamics (Guggenheimer, 2012; Fischer,1998; Nurkan, 2015; Angeles, 1998).

1 Dr. Öğr. Üyesi, Kırıkkale Üniversitesi, Mühendislik ve Doğa Bilimleri Fakültesi
Orcid: 0000 0001 9098 838X, faik.babadag@kku.edu.tr

https://doi.org/10.58830/ozgur.pub252.c1214

A dual number is defined by the form $a_0 + \varepsilon a_1$, where a_0 and a_1 are real numbers and ε is the dual unit taken to satisfy $\varepsilon^2=0$ with $\varepsilon \neq 0$.

The addition and multiplication of any dual numbers $\mathbb{A}$ and $\mathbb{B}$ are defined, respectively, as follows:

$\mathbb{A} + \mathbb{B} = (a_0 + b_0) + \varepsilon(a_1 + b_1)$

and

$\mathbb{A}\mathbb{B} = a_0 b_0 + \varepsilon(a_0 b_1 + a_1 b_0)$

In the literature, the Fibonacci and the Lucas numbers play an important role in various areas such as mathematics and related fields. For positive integer n, the linear sequences F_n and L_n are defined by $\mathrm{F}_{n+2}=\mathrm{F}_{n+1} + \mathrm{F}_n$ and $\mathrm{L}_{n+2}=\mathrm{L}_{n+1} + \mathrm{L}_n$. Here, the initial conditions are $\mathrm{F}_0=0$, $\mathrm{F}_1=1$, $\mathrm{L}_0=2$, and $\mathrm{L}_1=1$, respectively. The Binet formulas of these numbers are

$$\mathrm{F}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$\mathrm{L}_n = \alpha^n + \beta^n$$

The properties, relations, results between Fibonacci and Lucas numbers can be found in (Horadam, 1963; Koshy, 2018, 2019; Maynez, 2016; Nalli, 2009; Oduol, 2020; Vorobiov, 1974). Quantum calculus is important in both physics and mathematics. In recent years, many researchers have become interested in quantum calculus, which occurs in a variety of mathematical fields of combinatorics and special functions. For understanding of this paper, we demonstrate definitions and facts from the quantum calculus. The q-integer (Kac, 2002; Kome, 2022; Stum, 2013; Akkus, 2019) is shwon by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1} \qquad (1)$$

of two variables n and q. For positive integer m and n, we obtain

$$\begin{cases} [m+n]_q = [m]_q + q^m[n]_q \\ [mn]_q = [m]_q + [n]_{q^m} \end{cases} \tag{2}$$

In this paper, some basic concepts of dual number sequences are given using q-dual Fibonacci number sequences and q-dual Lucas number sequences. Moreover, we obtain the main identities and define quantum dual Polynomials and functions.

2. q -Dual Fibonacci and Lucas Number Sequences

In this section, we give dual number sequences with components including quantum integers, which are called q-dual Fibonacci number sequences and q-dual Lucas number sequences. Moreover, we obtain the main identities.

Definition 2.1. Dual number sequences can be given in the forms:

$$\mathcal{F}_n = \frac{\alpha^n(1-q^n)}{\alpha - \alpha q} + \varepsilon \frac{\alpha^{n+1}(1-q^{n+1})}{\alpha - \alpha q} \tag{3}$$

or equivalent

$$\mathcal{F}_n = \alpha^{n-1}[n]_q + \varepsilon \alpha^n [n+1]_q \tag{4}$$

are called the n^{th} q-dual Fibonacci number sequences and

$$\mathcal{L}_n = \frac{\alpha^n(1-q^{2n})}{1-q^n} + \varepsilon \frac{\alpha^{n+1}(1-q^{2n+2})}{1-q^{n+1}} \tag{5}$$

or equivalent

$$\mathcal{L}_n = \frac{\alpha^n[2n]_q}{[n]_q} + \varepsilon \frac{\alpha^{n+1}[2n+2]_q}{[n+1]_q} \tag{6}$$

are called as the n^{th} q-dual Lucas number sequences, where ε is the dual unit and $\varepsilon^2 = 0$.

Theorem 2.2. (Binet's formula) For $n \geq 0$, the Binet formulas for dual number sequences are

$$\begin{cases} \mathcal{F}_n = \alpha^{n-1}[n]_q\underline{\alpha} + (\alpha q)^n\underline{\beta} \\ \mathcal{L}_n = \dfrac{\alpha^n[2n]_q}{[n]_q}\underline{\gamma} + \alpha^{n+1}(1-q)\underline{\beta} \end{cases} \tag{7}$$

Moreover, these dual number sequences are shown another expression of the form

$$\begin{cases} \mathcal{F}_n = \dfrac{\alpha^n\underline{\alpha} - (\alpha q)^n\underline{\gamma}}{\alpha - \alpha q} \\ \mathcal{L}_n = \alpha^n\underline{\alpha} + (\alpha q)^n\underline{\gamma} \end{cases} \tag{8}$$

where

$$\begin{cases} \underline{\alpha} = 1 + \alpha\varepsilon \\ \underline{\beta} = \varepsilon \\ \underline{\gamma} = 1 + (\alpha q)\varepsilon \end{cases} \tag{9}$$

Proof. By taking Equalities 1, 2, 4 and 9, we have

$$\begin{aligned} \mathcal{F}_n &= \alpha^{n-1}[n]_q + \varepsilon\alpha^n[n+1]_q \\ &= \alpha^{n-1}[n]_q + \varepsilon\alpha^n\big([n]_q + q^n\big) \\ &= \alpha^{n-1}[n]_q(1+\alpha\varepsilon) + \varepsilon\alpha^n q^n \\ &= \alpha^{n-1}[n]_q\underline{\alpha} + (\alpha q)^n\underline{\beta} \end{aligned}$$

Calculate the Binet formula for the q-dual Lucas number sequences $\mathcal{L}_n$ in other form

$$\begin{aligned} \mathcal{L}_n &= \frac{\alpha^n(1-q^{2n})}{1-q^n} + \varepsilon\frac{\alpha^{n+1}(1-q^{2n+2})}{1-q^{n+1}} \\ &= \alpha^n(1+q^n) + \varepsilon\alpha^{n+1}(1+q^{n+1}) \\ &= \alpha^n(1+\alpha\varepsilon) + (\alpha q)^n(1+\alpha q\varepsilon) \\ &= \alpha^n\underline{\alpha} + (\alpha q)^n\underline{\gamma} \end{aligned}$$

Theorem 2.3. (Catalan's identity) For $n > r,\ r \geq 1$

$$\mathcal{F}_{n+r}\mathcal{F}_{n-r}-\mathcal{F}_n^2=\frac{\alpha^{2n-2}q^n}{1-q}\left([-r]_q+[r]_q\right)\left(1+[2]_q\varepsilon\right)$$

Proof. By using Equalities 1, 2, 8, and 9, after some calculations. We have

$$\begin{aligned}
&\mathcal{F}_{n+r}\mathcal{F}_{n-r}-\mathcal{F}_n^2\\
&\quad=\frac{\alpha^{n+r}\underline{\alpha}-(\alpha q)^{n+r}\underline{\gamma}}{\alpha-\alpha q}\ \frac{\alpha^{n-r}\underline{\alpha}-(\alpha q)^{n-r}\underline{\gamma}}{\alpha-\alpha q}-\left(\frac{\alpha^{n}\underline{\alpha}-(\alpha q)^{n}\underline{\gamma}}{\alpha-\alpha q}\right)^2\\
&\quad=\frac{\alpha^{2n-2}q^n\ (2-q^{-r}-q^r)}{(1-q)^2}\ \underline{\alpha}\,\underline{\gamma}\\
&\quad=\frac{\alpha^{2n-2}q^n\ (1+1-q^{-r}-q^r)}{(1-q)^2}\left(1+[2]_q\varepsilon\right)\\
&\quad=\frac{\alpha^{2n-2}q^n}{1-q}\left([-r]_q+[r]_q\right)\left(1+[2]_q\varepsilon\right)
\end{aligned}$$

This completes the proof.

Remember that in the case $r = 1$ in Theorem 2.3, it reduces to Cassini's identity of the q-dual Fibonacci number sequences.

Corollary 2.4. (Cassini identity) For $n \geq 1$, we will have

$$\mathcal{F}_{n+1}\mathcal{F}_{n-1}-\mathcal{F}_n^2=\frac{\alpha^{2n-2}q^n}{1-q}\left([-1]_q+[1]_q\right)\left(1+[2]_q\varepsilon\right)$$

Theorem 2.5. (d' Ocagne's identity) For any integer m and n, we have

$$\mathcal{F}_m\mathcal{F}_{n+1}-\mathcal{F}_n\mathcal{F}_{m+1}=\alpha^{m+n+1}\left([m]_q-[n]_q\right)\underline{\alpha}^2\left(1+\underline{\beta}\right)$$

Proof. Using Equalities 2, 3, 7 and 9, we get

$$\begin{aligned}
&\mathcal{F}_m\mathcal{F}_{n+1}-\mathcal{F}_n\mathcal{F}_{m+1}\\
&\quad=\alpha^{m-1}\left([m]_q\,\underline{\alpha}+(\alpha q)^m\underline{\beta}\right)\left(\alpha^n[n]_q\,\underline{\alpha}+(\alpha q)^{n+1}\underline{\beta}\right)\\
&\qquad-\alpha^{n-1}\left([n]_q\,\underline{\alpha}+(\alpha q)^n\underline{\beta}\right)\left(\alpha^m[m]_q\,\underline{\alpha}+(\alpha q)^{m+1}\underline{\beta}\right)\\
&\quad=\alpha^{m+n-1}\left(\frac{q^n-q^m}{1-q}\right)\underline{\alpha}^2+\alpha^{m+n}(q^m-q^n)\underline{\alpha\beta}\\
&\quad=\frac{\alpha^{m+n}}{\alpha}\left([m]_q-[n]_q\right)\underline{\alpha}^2+\alpha^{m+n}\left([n]_q-[m]_q\right)\underline{\alpha\beta}\\
&\quad=\frac{\alpha^{m+n}}{\alpha}\left([m]_q-[n]_q\right)\underline{\alpha}^2\left(1-\underline{\beta}\right)
\end{aligned}$$

Theorem 2.6. For positive integers m, r and s with $m \geq r$ and

$m \geq s$, then the following holds between the q-dual Fibonacci number sequences and the q-dual Lucas number sequences

$$\mathcal{L}_{m+r}\mathcal{F}_{m+s} - \mathcal{L}_{m+s}\mathcal{F}_{m+r} = 2q^m\alpha^{2m+r+s-1}\left([s]_q - [r]_q\right)\left(1+[2]_q\varepsilon\right)$$

Proof. By using Equalities 2, 3, 7 and 9, also doing necessary calculations, we will have

$$\begin{aligned}
&\mathcal{L}_{m+r}\mathcal{F}_{m+s} - \mathcal{L}_{m+s}\mathcal{F}_{m+r} \\
&= \left(\frac{\alpha^{m+r}[2m+2r]_q}{[m+r]_q}\underline{\gamma} + \alpha^{m+r+1}(1-q)\underline{\beta}\right)\left(\alpha^{m+s-1}[m+s]_q\,\underline{\alpha} + (\alpha q)^{m+s}\underline{\beta}\right) \\
&\qquad - \left(\frac{\alpha^{m+s}[2m+2s]_q}{[m+s]_q}\underline{\gamma} + \alpha^{m+s+1}(1-q)\underline{\beta}\right) \\
&= 2q^m\alpha^{2m+r+s-1}\left(\frac{q^r - q^s}{1-q}\right)\underline{\alpha\gamma} \\
&= 2q^m\alpha^{2m+r+s-1}\left(\frac{q^r - 1 + 1 - q^s}{1-q}\right)\underline{\alpha\gamma} \\
&= 2q^m\alpha^{2m+r+s-1}\left([s]_q - [r]_q\right)\left(1+[2]_q\varepsilon\right)
\end{aligned}$$

3. q- Dual Fibonacci and Lucas Polynomial Sequences

In this section, we define q-dual polynomial sequences which generalizes q-polynomial sequences $\mathrm{F}_n(t)$ and $\mathrm{L}_n(t)$. We obtain the Binet formulas for q-dual polynomial sequences. Moreover, we give some properties and identities for these quantum dual polynomial sequences

Definition 3.1. For $p(t)$ and $r(t)$ dual component polynomials, the q-polynomial sequences $\mathrm{F}_n(t)$ and $\mathrm{L}_n(t)$ are provided as follows:

$$\begin{cases} \mathrm{F}_{n+2}(t) = p(t)\mathrm{F}_{n+1}(t) - r(t)\mathrm{F}_n(t) \\ \mathrm{L}_{n+2}(t) = p(t)\mathrm{L}_{n+1}(t) - r(t)\mathrm{L}_n(t) \end{cases} \qquad (10)$$

Here, the initial conditions are $\mathrm{F}_0(t)$=0, $\mathrm{F}_1(t)$=1, $\mathrm{L}_0(t)$=2, and $\mathrm{L}_0(t)$=$p(t)$, respectively.

Clssify the q-dual polynomial squences $\mathrm{F}_n(t)$ and $\mathrm{L}_n(t)$
acording to the given the polynomials $p(t)$ and $r(t)$ values,
repectively.

1. Assume that $p(t) = aq + 1$ and $r(t) = a^2\, q$ are constant polynomials, we obtain as follows:

$$\begin{cases} \mathrm{F}_{n+2}(t) = (aq+1)\mathrm{F}_{n+1}(t) - a^2\, q\, \mathrm{F}_n(t) \\ \mathrm{L}_{n+2}(t) = (aq+1)\mathrm{L}_{n+1}(t) - a^2\, q\mathrm{L}_n(t) \end{cases}$$

2. Assume that $p(t) = \lambda(s)$ and $r(t) = -1$ are not constant polynomials, then we have

$$\mathrm{F}_{n+2}(t) = \lambda(s)\mathrm{F}_{n+1}(t) + \mathrm{F}_n(t)$$

wth the initial conditions $\mathrm{F}_0(t) = 0$, $\mathrm{F}_1(t) = 1$. From Equality 10, roots of $x^2 - p(t)x + r(t) = 0$ are

$$\alpha(x) = \frac{p(t) + \sqrt{p^2(t) - 4r(t)}}{2}$$

ad

$$\beta(x) = \frac{p(t) - \sqrt{p^2(t) - 4r(t)}}{2}$$

Then, the Binet formulas for q-polynomials $\mathrm{F}_n(t)$ and $\mathrm{L}_n(t)$ are

$$\mathrm{F}_n(t) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}$$

and

$$\mathrm{L}_n(t) = \alpha^n(x) + \beta^n(x)$$

Defintion 3.2. Let $\mathcal{F}_n(t)$ and $\mathcal{L}_n(t)$ be dual polynomial sequences Then, the folowing recurrence relations are obtained

$$\mathcal{F}_n = \mathrm{F}_n(t) + \varepsilon \mathrm{F}_{n+1}(t) \qquad (11)$$

and

$$\mathcal{L}_n(t) = \mathrm{L}_n(t) + \varepsilon \mathrm{L}_{n+1}(t) \qquad (12)$$

FromEquality 11, the initial conditions of the q-dual Fibonacci polynomial sequences are

$$\mathcal{F}_0 = \mathrm{F}_0(t) + \varepsilon \mathrm{F}_1(t) = \varepsilon$$

and

$\mathcal{F}_1 = \mathrm{F}_1(t) + \varepsilon \mathrm{F}_2(t) = 1 + p(t)$

From Euality 12, the initial conditions of the q-dual Lucas polynomial sequeces are

$\mathcal{L}_0(t) = \mathrm{L}_0(t) + \varepsilon \mathrm{L}_1(t) = 2 + \varepsilon p(t)$

and

$$\begin{aligned}\mathcal{L}_1(t) &= \mathrm{L}_1(t) + \varepsilon \mathrm{L}_2(t) \\ &= p(t) + \varepsilon\left(p^2(t) - r(t)\right)\end{aligned}$$

where ε is dual unit.

Theoem 3.3. The Binet formula of the q-dual polynomials $\mathcal{F}_n(t)$ and $\mathcal{L}_n(t)$ are

$$\mathcal{F}_n(t) = \frac{\alpha^n(x)\underline{\alpha}(x) - \beta^n(x)\underline{\beta}(x)}{\alpha(x) - \beta(x)}$$

and

$$\mathcal{L}_n(t) = \alpha^n(x)\underline{\alpha}(x) + \beta^n(x)\underline{\beta}(x)$$

Here,

$\underline{\alpha}(x) = 1 + \alpha(x)\varepsilon$

and

$\underline{\beta}(x) = 1 + \beta(x)\varepsilon$

Proof. For the q-dual polynomials $\mathcal{F}_n(t)$ and $\mathcal{L}_n(t)$, the proof is calculated similarly to the theorem 2.2.

By doing some calculations, the following relations can be obtained

$\mathcal{F}_1(t) - \alpha(x)\mathcal{F}_0(t) = \underline{\beta}(x)$

$\mathcal{F}_1(t) - \beta(x)\mathcal{F}_0(t) = \underline{\alpha}(x)$

$\mathcal{L}_1(t) - \alpha(x)\mathcal{L}_0(t) = (\beta(x) - \alpha(x))\underline{\beta}(x)$

$\mathcal{L}_1(t) - \beta(x)\mathcal{L}_0(t) = (\alpha(x) - \beta(x))\underline{\alpha}(x)$

4. q-Dual Fibonacci and Lucas Function Sequences

In this section, we define quantum dual function sequences or briefly q-dual function sequences in the quantum calculus.

Definition 4.1. Assume that $p(t)$ is an arbitrary function. Its q-derivative operator is shown by

$d_q p(t) = p(qt) - p(t)$ (Kac, 2002).

Note that in particular $d_q(t) = (q-1)t$,

$$\lim_{q\to 1} D_q p(t) = \lim_{q\to 1} \frac{p(qt) - p(t)}{(q-1)t} = \frac{d_q(t)}{dt} \qquad (13)$$

were $q \neq 1$. The n^{th} q-dual Fibonacci function sequences is given by $\mathcal{F}_n(t) = \mathrm{F}_n(t) + \mathrm{F}_{n+1}(t)\varepsilon$, where $\mathrm{F}_n(t)$ is the $n^{\,th}$ q-Fibonacci function sequences and ε is dual unit. Then, q-derivative is

$$\begin{cases} D_q\mathcal{F}_n(t) = D_q(\mathrm{F}_n(t) + \mathrm{F}_{n+1}(t)\varepsilon) \\ \qquad\qquad\;\; = D_q\mathrm{F}_n(t) + D_q\mathrm{F}_{n+1}(t)\varepsilon \end{cases} \qquad (14)$$

Here $D_q\mathrm{F}_n(t)$ demonstrate the derivative of $\mathrm{F}_n(t)$

Proposition 4.2. For the integer $n > 0$, if $\mathrm{F}_n(t) = (t - \mu)_q^n$ is selected, q-derivative of the function sequences $\mathrm{F}_n(t)$ can be given in the form

$D_q\mathrm{F}_n(t) = [n]_q\mathrm{F}_{n-1}(t)$,

where μ is constant.

Proof. From Equalities 13 and 14, we compute q-derivative of the function sequences $\mathrm{F}_n(t)$, we obtain

$$\begin{aligned} D_q\mathrm{F}_n(t) &= \frac{\big(q(t-\mu)\big)^n - (t-\mu)^n}{(q-1)(t-\mu)} \\ &= \frac{(q^n - 1)(t-\mu)^{n-1}}{(q-1)} \\ &= [n]_q\mathrm{F}_{n-1}(t) \end{aligned}$$

and the derivative of the $n^{\,th}$ q-dual Fibonacci function sequences is

$$\begin{aligned} D_q\mathcal{F}_n(t) &= [n]_q\mathrm{F}_{n-1}(t) + [n+1]_q\mathrm{F}_n(t)\varepsilon \\ &= [n]_q\mathrm{F}_{n-1}(t) + q^n[1]_q\,\varepsilon \end{aligned}$$

Example 4.3. For the integer $n < 0$, consider the function $\mathrm{F}_{-n}(t) = (t - \mu)_q^{-n}$, we calculate the q-derivative of $\mathrm{F}_n(t)$ and $\mathcal{F}_n(t)$.

Proof. By using Definition 3.1 and Equality 13, we can write

$$D_q\mathrm{F}_{-n}(t) = \frac{\left(q(t-\mu)\right)^{-n} - (t-\mu)^{-n}}{(q-1)(t-\mu)}$$

$$= \frac{(q^{-n}-1)(t-\mu)^{-n-1}}{(q-1)}$$

$$= [-n]_q \mathrm{F}_{-(n+1)}(t)$$

$$= -\frac{[n]_q}{q^n}\mathrm{F}_{-(n+1)}(t)$$

and doing necessary calculations, $D_q\mathcal{F}_{-n}(t)$, we get

$$D_q\mathcal{F}_{-n}(t) = -\frac{[n]_q}{q^n}\mathrm{F}_{-(n+1)}(t) - \frac{[n+1]_q}{q^{n+1}}\mathrm{F}_{-(n+2)}(t)\varepsilon$$

5. Conclusion

In the present paper, the q-dual number sequences have been introduced by using the notations from quantum calculus. First of all the recurrence relation for these numbers have been obtained. Then, some fundamental identities are obtained such as the Binet formulas, the Cassini, the Catalan, and the d'Ocagne identities. Furthermore, the new polynomials and functions which are called q-dual Fibonacci and q-dual Lucas polynomial and function sequences. Also, we have presented some properties and identities for these polynomial and function sequences.

References

[1] Kotelnikov, A. P. (1895). Screw Calculus and Some Applications to Geometry and Mechanics. Kazan, Russia: Annals of the Imperial University of Kazan.

[2] Study, E. (2022). Geometry der Dynamen. German edition, Legare Street Press, Berlin, Germany.

[3] Guggenheimer, H. W. (2012). Differential Geometry. Dover Publications, New York.

[4] Fischer, I. (1998). Dual Number Methods in Kinematics, Statics and Dynamics. New York, USA: CRC Press,.

[5] Nurkan, S. K., & Guven. I. A. (2015). Dual Fibonacci Quaternions, Advances in Applied Clifford Algebras, 25, 403–414.

[6] Angeles, J. (1998). The Application of Dual Algebra to Kinematic Analysis. Computational Methods in Mechanical Systems: Mechanism Analysis, Synthesis, and Optimization, NATO ASI Series, Springer Berlin Heidelberg, 161, 3–32.

[7] Horadam, A. F. (1961). A generalized Fibonacci sequence. The American Mathematical Monthly, 68(5), 455-459.

[8] Horadam, A. F. (1963). Complex Fibonacci numbers and Fibonacci quaternions. The American Mathematical Monthly, 70(3), 289–291.

[9] Koshy, T. (2018). Fibonacci and Lucas Numbers with Applications. 1, John Wiley Sons.

[10] Koshy, T. (2019). Fibonacci and Lucas Numbers with Applications, Volume 2, John Wiley Sons.

[11] Maynez, A. G., & Acosta A. P. (2016). A Method to Construct Generalized Fibonacci Sequences, Journal of Applied Mathematics, 2016, ID 4971594, http://dx.doi.org/10.1155/2016/4971594.

[12] Nalli, A., & Haukkanen, P. (2009). On Generalized Fibonacci and Lucas Polynomials, Chaos Solitons Fractals, 42, 3179-3186.

[13] Oduol, F., & Okoth, I. O. (2020). On generalized Fibonacci numbers. Communications in Advanced Mathematical Sciences, 3(4), 186–202.

[14] Vorobiov, N. (1974). Numeros De Fibonacci, Editorial MIR, Moscu, URSS.

[15] Kac, V., & Cheung, P. (2002). Quantum Calculus, Springer.

[16] Kome, S., Kome, C. & Catarino, P. (2022). Quantum Calculus Approach to the Dual Bicomplex Fibonacci and Lucas Numbers. Journal of Mathematical Extension, 6(2), 1-17.

[17] Stum, B., & Quiros, A. (2013). On Quantum Integers and Rationals, Hal Open Science 649/13022, 107-130.

[18] Akkus, I. & Kızılaslan, G. (2019). Quaternions: Quantum Calculus Approach with Applications, Kuwait Journal of Science 46(4), 1-13.