Chapter 2

Fractional Trigonometric Korovkin Theory Via Statistical Convergence With Respect To Power Series Method

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Abstract

In approximation theory, Korovkin-type theorems are well used since they provide us to determine the uniform convergence of positive linear operators to identity by using only three functions $\{1, x, x^2\}$. They have been investigated in different function spaces, generally, by using different concepts of convergences, by using q-calculus and rarely fractional calculus. In this chapter, by fractional calculus which is a branch of analysis dealing with derivatives and integrals of arbitrary order, fractional Korovkin-type trigonometric approximation results will be presented via P-statistical convergence which depends on a power series method. Also, as an application of our theorems various type examples will be constructed.

1. Introduction and Preliminaries

Weierstrass theorem which has a complicated proof deals with the approximation of algebraic and trigonometric polynomials to a continuous function on a closed interval and it has a key role in the development of approximation theory [33]. Since this proof is hard to follow, many mathematicians aim to give a simpler alternative proof. One of these mathematicians is Bernstein who has presented a short, smart proof by introducing Bernstein polynomials [8], [22]. Then this proof has been

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extended not only for positive linear but also nonlinear operators [6], [7], [10], [13], [21], [24], [27]. The limit used in these approximation theorems is the classical limit of operators. But what if the classical limit fails? In the case that the classical limit fails, different kinds of convergences have been introduced [4], [5], [14], [16], [30], [32]. One of these concepts is statistical convergence and the main motivation behind it is to replace finite sets of indices in ordinary convergence with sets of density zero. These concepts are effective to use since they generalize the ordinary convergence. Therefore, they have been considered in probability, measure and number theory, optimization, summability, trigonometric series and also approximation theory which has significant applications in polynomial approximation, functional analysis, differential and integral equations [18]. Recently, approximation theory has also been used in feedforward neural networks (FFNs), ReLU networks and deep learning which depends on structured deep neural networks [25], [31]. These results have successful applications in many areas of science and technology. Therefore, it is important to make contributions to the existing literature of approximation theory, especially Korovkin-type approximation theory.

Fractional calculus is the branch of analysis which deals with the investigation of integrals and derivatives of arbitrary order. Of course, integrals and derivatives are the fundamental concepts of analysis and it is interesting to wonder the non-integer order derivative of a function. Indeed, fractional calculus has a long mathematical history which has started with a letter between Leibniz and L'Hospital. The meaning of the derivative of $\frac{1}{2}$ order has been discussed in this letter and it has not taken into consideration enough up to Liouville, Grünwald, Letnikov and Riemann. Fractional derivatives have developed as a pure theoretical area of mathematics for three centuries but from the late 1900's, they have found practical applications in real world; for example it has been shown that fractional derivatives and integrals are very appropriate to describe the properties of polymers, rocks, different materials and processes. They also provide an important tool in physics, geology, earthquake dynamics, bioengineering, eloctromagnetic waves, mechanics [12], [26], [28]. Indeed, the existing mathematical theory of fractional calculus is behind the necessities of mathematical modellings of all these applications in real world. Therefore, it is important to investigate some results of analysis from the perspective of fractional calculus.

In the present chapter, the first aim is to examine fractional trigonometric Korovkin-type results for a sequence of positive linear operators with the use of statistical convergence depending on a power series, shortly P-statistical convergence. We also provide examples as an application of our theorems. We should point out that using P-statistical convergence in fractional Korovkin theory is the new idea of this study and it is important to note that there are a few papers which combine fractional calculus and approximation theory but very rare [1], [2], [3], [11], [19], [23].

Now we are ready to collect some basic notion, definitions and also the known results which will be needed along the paper.

The natural density of $G \subseteq \mathbb{N}$ is given by

$$\delta(G) := \lim_{k\to\infty} \frac{1}{k} \#\{n \le k : n \in G\}$$

if the limit exists where $\# E$ denotes the cardinality of E and $\mathbb{N}$ is the set of all natural numbers. If $\delta(G_\varepsilon) = 0$ for every $\varepsilon > 0$ where $G_\varepsilon = \{n \in \mathbb{N}: |s_n - l| \ge \varepsilon\}$ then $s = (s_n)$ is said to be statistically convergent to l [15], [17], [29]. Let (p_n) be sequence of real numbers such that for all $n \ge 2,\ p_n \ge 0$, and $p_1 > 0,\ p(t) := \sum_{n=1}^{\infty} p_n t^{n-1}$ with a radius of convergence $R \in (0, \infty]$. Then power series method is defined as follows:

Let also

$$C_p := \{ f: (-R, R) \to \mathbb{R} \mid \lim_{0<t\to R^-} \frac{1}{p(t)} f(t) \ exists \}$$

and

$$C_{p_p} := \left\{ s = (s_n) \mid p_s(t) := \sum_{n=1}^{\infty} p_n t^{n-1} s_n \ with\, radius\, of\ convergence \ge R\, and\ p_s \in C_p \right\},$$

$$P - \lim x = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} s_n$$

where the functional $P - \lim : C_{p_p} \to \mathbb{R}$ and then it is said that s is P $-$convergent [9], [20]. Consider the following example: let $s_n = (-1)^n$, for $n \ge 1,\ p_n = 1$ then

$R=1,\ p(t)=\dfrac{1}{1-t}$ and

$$\lim_{t\to 1^-}(1-t)\sum_{n=1}^{\infty} s_n t^{n-1}=0$$

which means that $s=(s_n)$ is P-convergent to 0 but s is not convergent. Therefore we can say that power series method is more effective.

If a convergent sequence is also P-convergent to same limit then it is said that P is regular and it is characterized by the following condition:

$$\lim_{t\to R^-}\frac{p_n t^n}{p(t)}=0,\ \text{for every } n\in \mathbb{N}$$

[9], [20].

By combining statistical convergence and power series, Ünver and Orhan [32] have recently introduced P-statistical convergence and have presented a Korovkin-type theorem for positive linear operators defined on $C[0,1]$, the space of all continuous functions on $[0,1]$.

Now we are ready to recall P-statistical convergence and Caputo derivative which are the main tools of our results.
Let P be regular and $G\subseteq \mathbb{N}$. If

$$\delta_p(G):=\lim_{0<t\to R^-}\frac{1}{p(t)}\sum_{n\in G} p_n t^{n-1}$$

exists then it is said to be the P-density of G. One can immediately observe that if $\delta_p(G)$ exists then $\delta_p(G)\in[0,1]$ [32].

Let $s=(s_n)$ be a sequence of real numbers and let P be regular. If for every $\varepsilon>0$

$$\lim_{0<t\to R^-}\frac{1}{p(t)}\sum_{n\in G_\varepsilon} p_n t^{n-1}=0$$

that is, $\delta_p(G_\varepsilon)=0$ for every $\varepsilon>0$, then s is said to be P-statistically convergent to l and we denote it by $st_p-\lim s_n=l$ [32].

In order to show that statistical convergence does not imply P-statistical convergence and vice versa, illustrative examples have been provided in [32]. Therefore we can emphasize that our results make important contribution to the existing literature.

There are many possible generalizations of $\frac{d^n}{dx^n} f(x)$ in the case of $n \notin \mathbb{N}$. Some of them are Riemann-Liouville, Caputo, Grünwald-Letnikov, Weyl, Riesz and they are well studied by many mathematicians. These different definitions of fractional derivatives give the oppurtunity to study with the most suitable one with the problem and to obtain the best solution. Although there are important relations between these definitions, the physical meanings of them differ from each other. For example, an attractive difference between Riemann-Liouville and Caputo is the derivative of a constant. The Caputo derivative of a constant is zero but for a finite lower bound Riemann-Liouville derivative is not zero. In order to give a physical comment of a problem, it is necessary to be satisfied that the derivative of a constant is zero. Hence, Caputo derivatives are well used in the existing literature and here we consider them.

Throughout the paper we consider the closed interval $J = [-\pi, \pi]$, we let μ be a positive real number, m is the ceiling of the number μ, i.e., $m = \lceil \mu \rceil$, Γ is the Gamma function and $AC(J) = \{ f : J \to \mathbb{R}, f \textit{ is absolutely continuous} \}$,

$$AC^m(J) = \{ f : J \to \mathbb{R}, f^{(m-1)} \in AC(J) \}.$$

Then, the left and right Caputo fractional derivatives of $f \in AC^m(J)$ are defined by

$$D^{\mu}_{*(-\pi)} f(y) := \frac{1}{\Gamma(m-\mu)} \int_{-\pi}^{y} (y-t)^{m-\mu-1} f^{(m)}(t)\, dt$$

for $y \in J$,

$$D^{\mu}_{\pi^-} f(y) := \frac{(-1)^m}{\Gamma(m-\mu)} \int_{y}^{\pi} (\xi - y)^{m-\mu-1} f^{(m)}(\xi)\, d\xi$$

for $y \in J$, respectively. Here, we also let $D^{0}_{*(-\pi)} f = f$, $D^{0}_{\pi^-} f = f$ on J and suppose for $y < -\pi, D^{\mu}_{*(-\pi)} f(y) = 0$ and for $y > \pi, D^{\mu}_{\pi^-} f(y) = 0.$ The followings are well known from [1], [2], [3].

1. If $\mu > 0,\ \mu \notin \mathbb{N},\ m = \lceil \mu \rceil,\ f \in C^{m-1}(J)$ and $f^{(m)} \in L_{\infty}(J)$ then $D^{\mu}_{*(-\pi)} f(-\pi) = 0,\ D^{\mu}_{\pi^-} f(\pi) = 0.$

2. Let $y \in J$ be fixed. For $\mu > 0,\ m = \lceil \mu \rceil,\ f \in C^{m-1}(J)$ and $f \in L_{\infty}(J)$, take into consideration the following Caputo fractional derivatives:

$$U_f(x,y) := D^{\mu}_{*x} f(y) := \frac{1}{\Gamma(m-\mu)} \int_x^y (y-t)^{m-\mu-1} f^{(m)}(t)\,dt,\ for\ y \in [x,\pi]$$

and

$$V_f(x,y) := D^{\mu}_{x^-} f(y) := \frac{(-1)^m}{\Gamma(m-\mu)} \int_y^x (\xi - y)^{m-\mu-1} f^{(m)}(\xi)\,d\xi,\ for\ y \in [-\pi, x].$$

Then for each fixed $x \in J, U_f(x,.)$ and $V_f(x,.)$ are continuous on $[x,\pi]$ and $[-\pi,x]$, respectively. Furthermore, $U_f(.,.), V_f(.,.)$ are continuous on $J \times J$ in the case $f \in C^m(J)$.

3. If $g \in C(J \times J)$, then $h(x) := w(g(x,.),\delta)_{[-\pi,x]}$ and $r(x) := w(g(x,.),\delta)_{[x,\pi]}$ are continuous for any $\delta > 0$ at the point $x \in J$ where $w(f,\delta),\ \delta > 0$ is the modulus of continuity.

4. For any $\delta > 0$,

$$\sup_{x \in J} w(U_f(x,.),\delta)_{[x,\pi]} < \infty$$

and

$$\sup_{x\in J} w\left(V_f\left(x,.\right),\delta\right)_{[-\pi,x]} < \infty$$

if $f \in C^{m-1}\left(J\right)$ with $f^{(m)} \in L_\infty\left(J\right)$.

5. By setting $\rho_{n,\mu} := \left\| T_n\left(\phi^{\mu+1}\right)\right\|^{\frac{1}{\mu+1}}$, we can write

$$\left\| T_n\left(f\right) - f\right\| \leq K_{m,\mu}\left\{\left\| T_n\left(e_0\right) - e_0\right\| + \sum_{k=1}^{m-1}\left\| T_n\left(\left|\psi\right|^k\right)\right\|\right.$$

$$+\rho_{n,\mu}^{\mu}\left(\sup_{x\in J} w\left(U_f\left(x,.\right),\rho_{n,\mu}\right)_{[x,\pi]}\right) + \rho_{n,\mu}^{\mu}\left(\sup_{x\in J} w\left(V_f\left(x,.\right),\rho_{n,\mu}\right)_{[-\pi,x]}\right)$$

$$\left.+\rho_{n,\mu}^{\mu}\left\| T_n\left(e_0\right) - e_0\right\|^{\frac{1}{\mu+1}}\left(\sup_{x\in J} w\left(U_f\left(x,.\right),\rho_{n,\mu}\right)_{[x,\pi]} + \sup_{x\in J} w\left(V_f\left(x,.\right),\rho_{n,\mu}\right)_{[-\pi,x]}\right)\right\}$$

where

$$K_{\mu,m} = \max\left\{\frac{\left(2\pi\right)^{\mu}}{\Gamma\left(\mu+1\right)}, \frac{\left(2\pi\right)^{\mu}\left(\mu+1+2\pi\right)}{\Gamma\left(\mu+2\right)}, \|f\|, \|f'\|, \frac{\|f''\|}{2!}, \ldots, \frac{\left\|f^{(m-1)}\right\|}{(m-1)!}\right\},$$

$\psi\left(y\right) := \psi_x\left(y\right) = y - x,\ \phi\left(y\right) := \phi_x\left(y\right) = \sin\left(\frac{\left|y-x\right|}{4}\right)$ and $e_0\left(y\right) = 1$ on J and T_n are positive linear operators from $C\left(J\right)$ into $C\left(J\right)$. Notice that the sum in the above inequality collapses if $\mu \in \left(0,1\right)$.

2. Fractional Calculus and P-statistical Convergence

In this section, we present our main results which deal with the fractional trigonometric approximation in different function spaces by P-statistical convergence. Throughout the section,

we let $\mu > 0,\ \mu \notin \mathbb{N},\ m = \lceil \mu \rceil$.

Theorem 1. ([2]) Let $T_n : C(J) \to C(J)$ be positive linear operators. If the sequence $\rho_{n,\mu}$ converges to 0 as $n \to \infty$ and $\{T_n(e_0)\}$ is uniformly convergent to e_0 on J, then $\{T_n(f)\}$ converges uniformly to f on J for every $f \in AC^m(J)$ with $f^{(m)} \in L_\infty(J)$. Also, this uniform convergence is still true on J when $f \in C^m(J)$.

In order to obtain P-statistical version of Theorem 1, we first need the following lemma.

Lemma 1. Let P be regular and $T_n : C(J) \to C(J)$ be positive linear operators. If $st_P - \lim \|T_n(e_0) - e_0\| = 0$ and $st_P - \lim \rho_{n,\mu} = 0$ then $st_P - \lim \left\| T_n\left(|\psi|^k\right)\right\| = 0$ for every $k = 1, 2, \dots, m-1$.

Proof. Let $k \in \{1, 2, \dots, m-1\}$ be fixed. With the use of Hölder inequality for positive linear operators which has been obtained in [24],

$$\left\| T_n\left(|\psi|^k\right)\right\| \le (2\pi)^k \left[\left(\rho_{n,\mu}^k\right) \|T_n(e_0) - e_0\|^{\frac{\mu+1-k}{\mu+1}} + \left(\rho_{n,\mu}^k\right) \right]$$

has been obtained in [3]. Now let us define the following sets:

$$G = \{ n \in \mathbb{N}: \| T_n(|\psi|^k) \| \ge \varepsilon \}$$

$$G_1 = \{ n \in \mathbb{N}: \left(\rho_{n,\mu}\right)^k \| T_n(e_0) - e_0 \|^{\frac{\mu+1-k}{\mu+1}} \ge \frac{\varepsilon}{2(2\pi)^k} \}$$

$$G_2 = \{ n \in \mathbb{N}: \rho_{n,\mu} \ge \frac{1}{2\pi} \left(\frac{\varepsilon}{2}\right)^{\frac{1}{k}} \}.$$

Then it is immediate that $G \subseteq G_1 \cup G_2$. Also define the following sets:

$$G_1' = \{ n \in \mathbb{N}: \rho_{n,\mu} \ge \frac{1}{\sqrt{2\pi}} \left(\frac{\varepsilon}{2}\right)^{\frac{1}{2k}} \}$$

$$G_1'' = \{ n \in \mathbb{N}: \| T_n(e_0) - e_0 \| \ge \left(\frac{\varepsilon}{2(2\pi)^k}\right)^{\frac{\mu+1}{2(\mu+1-k)}} \}.$$

Then it follows that $G \subseteq G'_1 \cup G''_1 \cup G_2$.

Hence,

$$\frac{1}{p(t)}\sum_{n\in G} p_n t^{n-1} \le \frac{1}{p(t)}\sum_{n\in G'_1} p_n t^{n-1} + \frac{1}{p(t)}\sum_{n\in G''_1} p_n t^{n-1} + \frac{1}{p(t)}\sum_{n\in G_2} p_n t^{n-1}$$

holds and

$st_P - \lim \left\| T_n\left(|\psi|^k\right)\right\| = 0,$ for each $k = 1, 2, ..., m-1$ by the hypotheses. Thus we complete the proof. □

Now we can present the first fractional approximation result via P-statistical convergence.

Theorem 2. Let P be regular and $T_n : C(J) \to C(J)$ be positive linear operators. If $st_P - \lim \left\| T_n(e_0) - e_0 \right\| = 0$ and $st_P - \lim \rho_{n,\mu} = 0$ then $st_P - \lim \left\| T_n(f) - f \right\| = 0$ for every $f \in AC^m(J)$ such that $f^{(m)} \in L_\infty(J)$.

Proof. Let $f \in AC^m(J)$ with $f^{(m)} \in L_\infty(J)$. It is known that

$$\left\| T_n(f) - f \right\| \le H_{m,\mu}\left\{ \left\| T_n(e_0) - e_0 \right\| + \sum_{k=1}^{m-1} \left\| T_n\left(|\psi|^k\right)\right\| + 2\rho_{n,\mu}^{\mu} + 2\rho_{n,\mu}^{\mu} \left\| T_n(e_0) - e_0 \right\|^{\frac{1}{\mu+1}} \right\},$$

where

$$H_{m,\mu} = \max\left\{ K_{m,\mu}, \sup_{x\in J} w\left(U_f(x,.), \rho_{n,\mu}\right)_{[x,\pi]}, \sup_{x\in J} w\left(V_f(x,.), \rho_{n,\mu}\right)_{[-\pi,x]} \right\}.$$

Again define the followings:

$$F = \{ n \in \mathbb{N} : \| T_n(f) - f \| \ge \varepsilon \}$$

$$F_k = \left\{ n \in \mathbb{N} : \| T_n(|\psi|^k) \| \ge \frac{\varepsilon}{(m+2)H_{m,\mu}} \right\}, k = 1, 2, \dots, m-1$$

$$F_m = \{ n \in \mathbb{N} : \| T_n(e_0) - e_0 \| \ge \frac{\varepsilon}{(m+2)H_{m,\mu}} \}$$

$$F_{m+1} = \{ n \in \mathbb{N} : \rho_{n,\mu} \geq \left(\frac{\varepsilon}{2(m+2)H_{m,\mu}} \right)^{\frac{1}{\mu}} \}$$

$$F_{m+2} = \left\{ n \in \mathbb{N} : \rho_{n,\mu}^{\mu} \parallel T_n(e_0) - e_0 \parallel^{\frac{1}{\mu+1}} \geq \frac{\varepsilon}{2(m+2)H_{m,\mu}} \right\}.$$

Then it follows that $F \subseteq \bigcup_{i=1}^{m+2} F_i$. If we also define the following sets,

$$F_{m+3} = \{ n \in \mathbb{N} : \parallel T_n(e_0) - e_0 \parallel \geq \left(\frac{\varepsilon}{2(m+2)H_{m,\mu}} \right)^{\frac{\mu+1}{2}} \}$$

and

$$F_{m+4} = \{ n \in \mathbb{N} : \rho_{n,\mu} \geq \left(\frac{\varepsilon}{2(m+2)H_{m,\mu}} \right)^{\frac{1}{2\mu}} \}$$

then we have $F_{m+2} \subseteq F_{m+3} \cup F_{m+4}$ and $F \subseteq \bigcup_{i=1}^{m+4} F_i$. From the hypotheses, we obtain $\delta_P\left(F\right) = 0$ and this completes the proof. □

If we consider $C^m\left(J\right)$ instead of $AC^m\left(J\right)$, then we can slightly modify the above theorem. For this, let us prove the next lemma.

Lemma 2. Let P be regular and $T_n : C\left(J\right) \to C\left(J\right)$ be positive linear operators. If $st_P - \lim \rho_{n,\mu} = 0$ then we have

$$st_P - \lim \left(\sup_{x \in J} w\left(U_f\left(x,.\right), \rho_{n,\mu}\right)_{[x,\pi]} \right) = 0$$

and

$$st_P - \lim \left(\sup_{x \in J} w\left(V_f\left(x,.\right), \rho_{n,\mu}\right)_{[-\pi,x]} \right) = 0.$$

Proof. It is already known from [1], [2], that there exists $x_0, x_1 \in J$ such that

$$\sup_{x\in J} w\left(U_f(x,.),\rho_{n,\mu}\right)_{[x,\pi]} = w\left(U_f(x_0,.),\rho_{n,\mu}\right)_{[x_0,\pi]} =: p\left(\rho_{n,\mu}\right)$$

and

$$\sup_{x\in J} w\left(V_f(x,.),\rho_{n,\mu}\right)_{[-\pi,x]} = w\left(V_f(x_1,.),\rho_{n,\mu}\right)_{[-\pi,x_1]} =: q\left(\rho_{n,\mu}\right).$$

By the hypotheses, we get $\delta_P(\{ n \in \mathbb{N}: \rho_{n,\mu} \geq \delta \}) = 0$ for any $\delta > 0$. Then, by following the similar arguments in [3], we have that

$$\{ n \in \mathbb{N}: p(\rho_{n,\mu}) \geq \varepsilon \} \subseteq \{ n \in \mathbb{N}: \rho_{n,\mu} \geq \delta_1 \}$$

and

$$\{ n \in \mathbb{N}: q(\rho_{n,\mu}) \geq \varepsilon \} \subseteq \{ n \in \mathbb{N}: \rho_{n,\mu} \geq \delta_2 \}$$

which imply

$$\frac{1}{p(t)} \sum_{n:p\left(\rho_{n,\mu}\right)\geq\varepsilon} p_n t^{n-1} \leq \frac{1}{p(t)} \sum_{n:\rho_{n,\mu}\geq\delta_1} p_n t^{n-1},$$

$$\frac{1}{p(t)} \sum_{n:q\left(\rho_{n,\mu}\right)\geq\varepsilon} p_n t^{n-1} \leq \frac{1}{p(t)} \sum_{n:\rho_{n,\mu}\geq\delta_2} p_n t^{n-1}.$$

Then by taking limit in both sides and using the hypotheses, we complete the proof.

Now we can present the following result in $C^m(J)$. Since the technic of the proof is similar in earlier results, we omit the proof here.

Theorem 3. Let P be regular and $T_n : C(J) \to C(J)$ be positive linear operators. If $st_P - \lim\left\|T_n(e_0) - e_0\right\| = 0$ and $st_P - \lim \rho_{n,\mu} = 0$ then $st_P - \lim\left\|T_n(f) - f\right\| = 0$ for every $f \in C^m(J)$.

3. Applications

This section is devoted to the construction of special sequences of operators which support our results. Here, it is worthy to note that it is not possible to have approximation by earlier results. But we overcome this critical weakness of ordinary convergence and statistical convergence with the use of our method.

Example 1. Define the sequences (p_n) and (s_n) as follows:

$$p_n = \begin{cases} 1, & n = 2k \\ 0, & n = 2k+1 \end{cases}, \quad s_n = \begin{cases} 0, & n = 2k \\ \sqrt{n}, & n = 2k+1 \end{cases}.$$

One can obtain that the method P is regular and also observe that

$$K_\varepsilon = \{ n \in \mathbb{N} : |s_n - 0| \geq \varepsilon \} \subseteq \{ n = 2k+1 : k \in \mathbb{N} \}$$

holds for every $\varepsilon > 0$. Then we have

$$\delta_P(K_\varepsilon) = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{n\in K_\varepsilon} p_n t^{n-1} = 0$$

i.e., $st_P - \lim s_n = 0$. Let

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} f\left(-\pi + \frac{2\pi k}{n}\right)\left(\frac{x+\pi}{2\pi}\right)^k \left(\frac{\pi - x}{2\pi}\right)^{n-k},$$

$(s_n),(p_n)$ given above and define

$$T_n(f;x) = (1+s_n) B_n(f;x), \; x \in J, \; n \in \mathbb{N}.$$

Let $\mu = \frac{1}{2}$, $f \in AC^m(J)$ with $f^{(m)} \in L_\infty(J)$. Then
$m = \lceil \mu \rceil = 1$, $st_P - \lim \left\| T_n(e_0) - e_0 \right\| = 0$,

$$\left\| T_n\left(|\psi|^{\frac{3}{2}}\right) \right\| \leq \pi^{\frac{3}{2}} (1+s_n) \frac{1}{n^{\frac{3}{4}}}$$

and

$$\rho^{\frac{3}{2}}_{n,\frac{1}{2}} = \left\| T_n\left(\phi^{\frac{3}{2}}\right) \right\| \leq \frac{1}{8} \left\| T_n\left(|\psi|^{\frac{3}{2}}\right) \right\| \leq \pi^{\frac{3}{2}} (1+s_n) \frac{1}{8n^{\frac{3}{4}}}.$$

Then T_n satisfies the conditions of our theorem.

Example 2. Define the sequences $(p_n),(u_n)$ and (s_n) as follows:

$$p_n=\begin{cases}1, & n=2k\\ 0, & n=2k+1\end{cases},\quad u_n=\begin{cases}1, & n=2k\\ \dfrac{1}{2}, & n=2k+1\end{cases},\quad s_n=\begin{cases}0, & n=2k\\ \sqrt{n}, & n=2k+1\end{cases}.$$

One can obtain that the method P is regular and also observe that

$$K_\varepsilon=\{\,n\in\mathbb{N}: |s_n-0|\geq\varepsilon\,\}\subseteq\{\,n=2k+1: k\in\mathbb{N}\,\}$$

holds for every $\varepsilon>0$. Then we have

$$\delta_P(K_\varepsilon)=\lim_{0<t\to R^-}\frac{1}{p(t)}\sum_{n\in K_\varepsilon}p_n t^{n-1}=0$$

i.e., $st_P-\lim s_n=0$, $st_P-\lim u_n=1$.

Also define

$$T_n(f;x)=(1+s_n)\sum_{k=0}^{n}\binom{n}{k}f\left(-\pi+\frac{2\pi k}{n}\right)\left(\frac{u_n x+\pi}{2\pi}\right)^k\left(\frac{\pi-u_n x}{2\pi}\right)^{n-k},\ x\in J,\ n\in\mathbb{N}.$$

Let $\mu=\frac{1}{2}$, $f\in AC^m(J)$ with $f^{(m)}\in L_\infty(J)$. Then $m=\lceil\mu\rceil=1$, $st_P-\lim\|T_n(e_0)-e_0\|=0$,

$$\left\|T_n\left(|\psi|^{\frac{3}{2}}\right)\right\|\leq\pi^{\frac{3}{2}}(1+s_n)\left[(1-u_n)^2+\frac{1}{n}\right]^{\frac{3}{4}}$$

and

$$\rho_{n,\frac{1}{2}}^{\frac{3}{2}}=\left\|T_n\left(\phi^{\frac{3}{2}}\right)\right\|\leq\frac{1}{8}\left\|T_n\left(|\psi|^{\frac{3}{2}}\right)\right\|\leq\pi^{\frac{3}{2}}(1+s_n)\frac{1}{8}\left[(1-u_n)^2+\frac{1}{n}\right]^{\frac{3}{4}}.$$

Then T_n satisfies the conditions of our theorem. Again it is not possible to approximate f by using $T_n(f)$ since the sequence (s_n) is not convergent or statistically convergent. Furthermore it is still possible to approximate f by using $T_n(f)$ for every $f\in AC^m(J)$ with

$f^{(m)} \in L_{\infty}(J)$ via $P-$statistical convergence since (s_n) is $P-$statistically convergent to 0.

Example 3. Construct T_n by

$$T_n(f;x) = s_n B_n(f;x)$$

where $B_n(f;x)$ is given in Example 1 and

$$s_n = \begin{cases} 0, & n = 2k \\ \sqrt{n}, & n = 2k+1 \end{cases}, \quad p_n = \begin{cases} 1, & n = 2k \\ 0, & n = 2k+1 \end{cases}.$$

One can obtain that the method P is regular and $st_P - \lim s_n = 1$. As in the earlier examples, we notice that T_n satisfies our conditions for $\mu = \frac{1}{2}$, $f \in AC^m(J)$ with $f^{(m)} \in L_{\infty}(J)$.

Example 4. Define the sequences $(p_n), (u_n)$ and (s_n) as follows:

$$p_n = \begin{cases} 1, & n = 2k \\ 0, & n = 2k+1 \end{cases}, \quad u_n = \begin{cases} 1, & n = 2k \\ \frac{1}{2}, & n = 2k+1 \end{cases}, \quad s_n = \begin{cases} 0, & n = 2k \\ \sqrt{n}, & n = 2k+1 \end{cases}.$$

One can obtain that the method P is regular and also observe that

$$K_{\varepsilon} = \{ n \in \mathbb{N} : |s_n - 0| \geq \varepsilon \} \subseteq \{ n = 2k+1 : k \in \mathbb{N} \}$$

holds for every $\varepsilon > 0$. Then we have

$$\delta_P(K_{\varepsilon}) = \lim_{0<t \to R^-} \frac{1}{p(t)} \sum_{n \in K_{\varepsilon}} p_n t^{n-1} = 0$$

i.e., $st_P - \lim s_n = 0$, $st_P - \lim u_n = 1$.

Then define

$$T_n(f;x) = (1+s_n) \sum_{k=0}^{n} \binom{n}{k} f\left(-\pi + \frac{2\pi k}{n}\right) \left(\frac{u_n x + \pi}{2\pi}\right)^k \left(\frac{\pi - u_n x}{2\pi}\right)^{n-k}, \; x \in J, \; n \in \mathbb{N}.$$

Now let $\mu=\frac{1}{4}$, $f\in AC(J)$ with $f'\in L_{\infty}(J)$. Then $m=\lceil \mu \rceil=1$, $st_P-\lim\left\|T_n(e_0)-e_0\right\|=0$, by applying Hölder inequality for $p=\frac{8}{5}$, $q=\frac{8}{3}$, we obtain

$$\left\|T_n\left(|\psi|^{\frac{5}{4}}\right)\right\|\leq \pi^{\frac{5}{4}}(1+s_n)\left[(1-u_n)^2+\frac{1}{n}\right]^{\frac{5}{8}}$$

and

$$\rho_{n,\frac{1}{4}}^{\frac{5}{4}}=\left\|T_n\left(\phi^{\frac{5}{4}}\right)\right\|\leq \frac{1}{4^{\frac{5}{4}}}\left\|T_n\left(|\psi|^{\frac{5}{4}}\right)\right\|\leq \pi^{\frac{5}{4}}(1+s_n)\frac{1}{4^{\frac{5}{4}}}\left[(1-u_n)^2+\frac{1}{n}\right]^{\frac{5}{8}}$$

which imply $st_P-\lim \rho_{n,\frac{1}{4}}=0$.

Then T_n satisfies the conditions of our theorem. Again it is not possible to approximate f by using $T_n(f)$ since the sequence (s_n) is not convergent or statistically convergent. Furthermore it is still possible to approximate f by using $T_n(f)$ for every $f\in AC^m(J)$ with $f^{(m)}\in L_{\infty}(J)$ via $P-$statistical convergence since (s_n) is $P-$statistically convergent to 0.

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