Bölüm 1

Results on the Summability of Spliced Sequences by Using Nonnegative Matrices

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Abstract

Summability theory which aims to assign meaningful values to divergent sequences and series is a fascinating branch of mathematical analysis. In this theory, spliced sequences that are combinations of sequences obtained by partitions are important tools to understand and analyze the characters of series.

In this chapter, some results on spliced sequences will be presented by nonnegative matrices. It is benefical to note that the class of such matrices is more general.

Introduction

In the realm of mathematical analysis, a fascinating branch known as summability theory delves into the study of sequences and series, aiming to assign meaningful values to potentially divergent mathematical expressions. Within this field, the concept of spliced sequences emerges as a powerful tool for understanding and analyzing the convergence behavior of series. Spliced sequences refer to the combination of given sequences. The concept of spliced sequences has been introduced by Osikiewicz [12] and then this concept has been studied by Ünver et al. [15] and Ünver [16] in topological spaces. Also Yurdakadim et al. [17] have generalized this concept by using bounded sequences instead of convergent sequences. In this exploration, we will delve into the fascinating world of spliced sequences within the realm of summability theory. We will explore the techniques and methodologies used to combine sequences, investigate their

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convergence properties and examine their applications in resolving issues related to divergent sequences. On the other hand spliced sequences have been studied from a different perspective in [1], [3], [4], [7]. In this chapter, we give some results about spliced sequences for a more general class of matrices. These matrices need not to be regular.

Now we remind some basic definitions which we need throughout the paper.

Definition 1. If

$$\delta(G) := \lim_{n\to\infty} \frac{1}{n} \#\{j \le n : j \in G\}$$

exists then it is said the natural density of subset $G \subset \mathbb{N}$ where $\#(G)$ denotes the cardinality of G. The foundations of the concept of density are laid in [5], [6] and [9].

A sequence $s=(s_j)$ is called statistically convergent to L if for every $\varepsilon>0$

$$\lim_{n\to\infty} \frac{1}{n} \#\{j \le n : |s_j - l| \ge \varepsilon\} = 0$$

that is, $\delta(\{j \le n : j \in G_\varepsilon\})=0$ for every $\varepsilon > 0$ where $G_\varepsilon = \{j \in \mathbb{N} : |s_j - l| \ge \varepsilon\}$ [8], [10], [11], [14].

Definition 2. An R-partition of $\mathbb{N}$ is a set with finite number of infinite sets $P_i=\{v_i(k)\}$ for i=1,...,R such that $\bigcup_{i=1}^{R} P_i = \mathbb{N}$ and $P_i \cap P_j = \emptyset$ for all i≠j where R is a fixed positive integer [17].

Definition 3. An $\infty - partition$ on $\mathbb{N}$ is a set of countably infinite number of infinite sets $P_i=\{v_i(k)\}$ for $i \in \mathbb{N}$ such that $\bigcup_{i=1}^{\infty} P_i = \mathbb{N}$ and $P_i \cap P_j = \emptyset$ for all i≠j [17].

Definition 4. Let $s^{(i)} = \left(s_k^{(i)}\right)$ be a sequence in X with $\lim_{k\to\infty} s_k^{(i)} = \alpha_i$, i=1,...,R and $\{P_1, P_2, \dots, P_R\}$ be fixed R-partition. If $j \in P_i$, then $j=v_i(k)$ for some k. Define $s=(s_j)$ by $s_j = s_{v_i(k)} = s_k^{(i)}$. Then s is said to be an R-splice over $\{P_i : \mathrm{i} = 1, \dots, \mathrm{R}\}$ with limit points $\alpha_1, \alpha_2, \dots, \alpha_R$ [17].

Definition 5. Let $\{P_i : \mathrm{i} \in \mathbb{N}\}$ be a fixed infinite-partition of $\mathbb{N}$ and $s^{(i)} = \left(s_k^{(i)}\right)$ be a sequence with $\lim_{k\to\infty} s_k^{(i)} = \alpha_i$, $i \in \mathbb{N}$. If $j \in P_i$, then $j=v_i(k)$ for some k. Define $s=(s_j)$ by $s_j = s_{v_i(k)} = s_k^{(i)}$. Then it is said that s is an infinite-splice over $\{P_i : \mathrm{i} \in \mathbb{N}\}$ with limit points $\alpha_1, \alpha_2, \dots, \alpha_R, \dots$[17].

Example 1. If R=3, take the partition $P_1 = \{3k-1: k \in \mathbb{N}\}$, $P_2 = \{3k-2: k \in \mathbb{N}\}$ and $P_3 = \{3k: k \in \mathbb{N}\}$ and consider three convergent sequences $s^{(1)}$, $s^{(2)}$ and $s^{(3)}$. The 3-splice s of $s^{(1)}$, $s^{(2)}$ and $s^{(3)}$ over the 3-partition $\{P_1, P_2, P_3\}$ can be expressed by

$$s_j = \begin{cases} s_k^{(1)}, & j = 3k-1 \\ s_k^{(2)}, & j = 3k-2 \\ s_k^{(3)}, & j = 3k \end{cases}$$

that is, s=$\{s_1^{(1)}, s_1^{(2)}, s_1^{(3)}, s_2^{(1)}, s_2^{(2)}, s_2^{(3)}, \dots\}$.

Example 2. If we consider an infinite-partition of $\mathbb{N}$ as follows: $\{ P_i : P_i = \{2^{i-1}(2k-1)\}_{k=1}^{\infty}\}$ and convergent sequences $(s_k^{(i)})$ for $i \in \mathbb{N}$. We can construct an infinite spliced sequence s as above.

Theorem 1. An infinite matrix H= (h_{nk}) is regular if and only if

i) $\sup_n \sum_k \lfloor h_{nk} \rfloor < \infty$
ii) $\lim_n \sum_k h_{nk} = 1$
iii) $h_k := \lim_n h_{nk} = 0$ for all k$\in \mathbb{N}$ [2].

Let H= (h_{nk}) be nonnegative and regular. If $\delta_H(G) := \lim_n \sum_{k \in G} h_{nk}$ exists then $\delta_H(G)$ is called H-density of $G \subset \mathbb{N}$.

A sequence s=(s_k) H-statistically converges to L if for every $\varepsilon > 0$, $\delta_H(G_\varepsilon)$=0.

Let H= (h_{nk}) be infinite matrix. The characteristic of H is defined by

$$\chi(H) := \lim_n \sum_k h_{nk} - \sum_k h_k$$

where the series converge and the limit exists and $\lim_n h_{nk} = h_k$. For a given conservative matrix H= (h_{nk}), it is known that $\chi(H)$ exists [2]. Let H be infinite matrix and let P= $\{v_j\}$ be an infinite subset of $\mathbb{N}$. Then the matrix $H^{[\mathrm{P}]} = (d_{nk})$ is said to be a column submatrix of H, where $d_{nk} = h_{n,v_k}$ for all $n, k \in \mathbb{N}$. From [13] it is well known that.

Theorem 2. Let H= (h_{nk}) be infinite matrix such that $\chi(H)$ is defined. If there exists an integer r such that $h_{nk} \geq 0$ for all k$\geq r$ then

$$\liminf_n (Hs)_n \geq \sum_{k=1}^{\infty} h_k s_k + \chi(H) \liminf_n s_n$$

and

$$\limsup_{n}(Hs)_n \leq \sum_{k=1}^{\infty} h_k\, s_k + \chi(H)\limsup_{n} s_n$$

whenever $\sum_{k=1}^{\infty} h_k\, s_k$ converges [13].

Throughout the paper, we consider nonnegative matrices H= (h_{nk}) satisfying

i) $\lim_{n\to\infty} h_{nk} = 0$ for all k;
ii) $\lim_{n} \sum_{k=1}^{\infty} h_{nk} < \infty$.

The last condition implies

$$\sup_{n} \sum_{k=1}^{\infty} h_{nk} = T < \infty.$$

Then we obtain

$$\sum_{P\in C} \delta_H(\mathrm{P}) \leq \mathrm{T}$$

where C:={P⊂ ℕ: $\delta_H(\mathrm{P}) > 0$}.

2. MAIN RESULTS

In this section, by using a more general class of matrices we give some results which have generalized one in [17].

Lemma 1. If $\delta_H(\mathrm{P})$ exists then

$$\liminf_{n}\left(H^{[\mathrm{P}]}s\right)_n \geq \delta_H(\mathrm{P})\liminf_{k} s_k \tag{2.1}$$

and

$$\limsup_{n}\left(H^{[\mathrm{P}]}s\right)_n \leq \delta_H(\mathrm{P})\limsup_{k} s_k \tag{2.2}$$

where H= (h_{nk}) is nonnegative infinite matrix, P:=$\{v_k\}$ is an infinite subset of ℕ and s=(s_k) is bounded sequence.

Proof. Since H is nonnegative, it is obvious that $d_k \coloneqq \lim_n d_{nk} \geq 0$ for all $k \in \mathbb{N}$, where $d_{nk} = h_{n,v_k}$ for all $n, k \in \mathbb{N}$. Then

$$\begin{aligned}\liminf_{n}\left(H^{[\mathrm{P}]}s\right)_n &\geq \sum_{k=1}^{\infty} d_k\, s_k + \chi\left(H^{[\mathrm{P}]}\right)\liminf_{k} s_k\\ &= \chi\left(H^{[\mathrm{P}]}\right)\liminf_{k} s_k\\ &=\left(\lim_n \textstyle\sum_k d_{nk} - \sum_k d_k\right)\liminf_{k} s_k\end{aligned}$$

$=(\lim_n \sum_k h_{n,v_k})\ \liminf_k s_k$

$=(\lim_n \sum_{k\in P} h_{nk})\ \liminf_k s_k$

$=\delta_H(\mathrm{P})\liminf_k s_k$

holds by Theorem 2. If we take -s instead of s in (2.1), we immediately obtain (2.2).

2.1. Finite Splices. Here, $finite^*$ spliced sequences will be reminded and the results about their summability will be examined with the help of nonnegative matrices.

Definition 6. If j∈ P_i, then j= $v_i(k)$ for some k. Define s=(s_j) as $s_j = s_{v_i(k)} = s_k^{(i)}$. Then s is called an $R^* -$ splice over $\{P_i\colon \mathrm{i} = 1,2,\dots,\mathrm{R}\}$ where $\{P_i\colon \mathrm{i} = 1,2,\dots,\mathrm{R}\}$ is a fixed R-partition of ℕ and $s^{(i)} = \left(s_k^{(i)}\right)$ are bounded sequences for i = 1,2, … , R [17].

Remark that spliced sequences are constructed from convergent sequences and every R-splice is also $R^* -$ splice. Also, any $R^* -$ splice is bounded.

With the use of next theorem, we can estimate the core of the sequence $(Hs)_n$.

Theorem 3. If $\delta_H(P_i)$ exists for all i = 1,2, … , R then for any $R^* -$ splice s over $\{P_i\}$ we have

$$\liminf_n (Hs)_n \geq \sum_{i=1}^{R} \delta_H(P_i)\alpha_i \qquad (2.3)$$

and

$$\limsup_n (Hs)_n \leq \sum_{i=1}^{R} \delta_H(P_i)\beta_i \qquad (2.4)$$

where $\alpha_i = \liminf_k s_k^{(i)}$, $\beta_i = \limsup_k s_k^{(i)}$, H is a nonnegative summability matrix and $\{P_i = \{v_i(j)\}\colon \mathrm{i} = 1,2,\dots,\mathrm{R}\}$ is an R-partition of ℕ.

Proof. Suppose that $\delta_H(P_i)$ exists for all i = 1,2, … , R and let s be an $R^* -$ splice over $\{P_i\}$. As in [12],

$$(Hs)_n = \sum_{k=1}^{\infty} h_{nk}\, s_k$$

$=\sum_{i=1}^{R}(\sum_{k\in P_i} h_{nk} s_k)$

$$=\sum_{i=1}^{R}(\sum_{j=1}^{\infty} h_{n,v_i(j)} s_{v_i(j)})$$

$$=\sum_{i=1}^{R}(\sum_{j=1}^{\infty} h_{n,v_i(j)}\, s_j^{(i)})$$

$$=\sum_{i=1}^{R}(H^{[P_i]}s^{(i)})_n \qquad (2.5)$$

holds for all $n \in \mathbb{N}$. Thus,

$$\liminf_n (Hs)_n = \liminf_n \sum_{i=1}^{R}\left(H^{[P_i]}s^{(i)}\right)_n$$

$$\geq \sum_{i=1}^{R} \liminf_n \left(H^{[P_i]}s^{(i)}\right)_n$$

$$\geq \sum_{i=1}^{R} \delta_H(P_i)\alpha_i$$

holds by (2.5) and Lemma 1. This completes the proof of (2.3).

If we take -s instead of s in (2.3), then we immediately obtain (2.4).

If $s^{(i)}$ is convergent for any $i = 1,2,\dots,R$ then $\theta_i \coloneqq \xi_i = \sigma_i$ for any $i = 1,2,\dots,R$.

Hence, so from Theorem 3, we obtain that the core of the sequence Hs lies in $[\sum_{i=1}^{R} \delta_H(P_i)\xi_i, \sum_{i=1}^{R} \delta_H(P_i)\sigma_i\,]$ and generalizes the similar theorems in [12], [17].

2.1. Infinite Splices. Here, $\infty^* -$ spliced sequences will be reminded and the results about their summability will be examined with the help of nonnegative matrices.

Definition 7. Let $\{P_i : i \in \mathbb{N}\}$ be a fixed infinite-partition of $\mathbb{N}$ and let $s^{(i)} = \left(s_k^{(i)}\right)$ be bounded sequences for $i \in \mathbb{N}$. If $j \in P_i$, then $j = v_i(k)$ for some k. Define $s = (s_j)$ as $s_j = s_{v_i(k)} = s_k^{(i)}$. Then it is said that s is an $\infty^* -$ splice over $\{P_i : \}$ [17].

Recall that the spliced sequences (∞-splice) are obtained from convergent sequences in [12] and again one can easily notice that any ∞-splice is also an $\infty^* -$ splice. Note that an $\infty^* -$ splice does not need to be bounded.

Now we deal with the core of the sequence *Hs* for a bounded $\infty^* -$ splice s in the following theorem.

Theorem 4. If $\delta_H(P_i)$ exists for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \delta_H(P_i) =$T, then for any bounded $\infty^* -$ splice s over $\{P_i\}$ we get

$$\liminf_n (Hs)_n \geq \sum_{i=1}^{\infty} \delta_H(P_i)\,\xi_i \tag{2.6}$$

and

$$\limsup_n (Hs)_n \leq \sum_{i=1}^{\infty} \delta_H(P_i)\sigma_i \tag{2.7}$$

where $\xi_i = \liminf_k s_k^{(i)}$, $\sigma_i = \limsup_k s_k^{(i)}$, H is a nonnegative infinite matrix and $\{P_i = \{v_i(j): i \in \mathbb{N}\}$ is an ∞-partition of $\mathbb{N}$.

Proof. Suppose that $\delta_H(P_i)$ exists for all $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} \delta_H(P_i) = T$ and let s be an ∞^* – splice s over $\{P_i\}$. Again, as in [12] that

$$\begin{aligned}(Hs)_n &= \sum_{k=1}^{\infty} h_{nk}\, s_k \\ &= \textstyle\sum_{i=1}^{\infty} \left(\sum_{k\in P_i} h_{nk} s_k\right) \\ &= \textstyle\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} h_{n,v_i(j)} s_{v_i(j)}\right) \\ &= \textstyle\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} h_{n,v_i(j)}\, s_j^{(i)}\right) \\ &= \textstyle\sum_{i=1}^{\infty} \left(H^{[P_i]} s^{(i)}\right)_n \end{aligned} \tag{2.8}$$

holds for all $n \in \mathbb{N}$. Set $\phi_n : \mathbb{N} \to \mathbb{C}$ and $\gamma_n : \mathbb{N} \to \mathbb{C}$ for every n by

$$\phi_n(i) := \left(H^{[P_i]} s^{(i)}\right)_n \text{ and } \gamma_n(i) := M\left(H^{[P_i]} e\right)_n$$

where $M := \sup_k \lfloor s_k \rfloor$ and $e = (1,1,\ldots)$. From Theorem 1.2 in [12] we know that

$$\lim_n \gamma_n(i) = M\,\delta_H(P_i)$$

and

$$\lim_n \int_{\mathbb{N}} \gamma_n(i)\, d\lambda = \int_{\mathbb{N}} \left(\lim_n \gamma_n(i)\right) d\lambda = MT > 0 \tag{2.9}$$

where λ is the counting measure. Also one can easily show that

$$\lfloor \phi_n(i) \rfloor \leq \gamma_n(i)$$

holds for all $n, i \in \mathbb{N}$. Since ϕ_n and γ_n are measurable with respect to λ and $\phi_n + \gamma_n \geq 0$ for all n, then it follows from (2.9) and Fatou's Lemma that

$$\lim_n \int_{\mathbb{N}} \left(\liminf_n (\phi_n + \gamma_n) \right) (\mathrm{i})\, d\lambda \leq \liminf_n \int_{\mathbb{N}} (\phi_n + \gamma_n)(\mathrm{i})\, d\lambda$$

$$= \liminf_n \left(\int_{\mathbb{N}} \phi_n(i) d\lambda + \int_{\mathbb{N}} \gamma_n(i) d\lambda \right)$$

$$= \liminf_n \int_{\mathbb{N}} \phi_n(i) d\lambda + \lim_n \int_{\mathbb{N}} \gamma_n(i) d\lambda$$

$$= \liminf_n \int_{\mathbb{N}} \phi_n(i) d\lambda + M \sum_{i=1}^{\infty} \delta_H(P_i)$$

$$= \liminf_n \int_{\mathbb{N}} \phi_n(i) d\lambda + MT. \qquad (2.10)$$

Beside this, (γ_n) is convergent and we have

$$\int_{\mathbb{N}} \liminf_n (\phi_n + \gamma_n)(i)\, d\lambda = \int_{\mathbb{N}} (\liminf_n \phi_n(i) + \lim_n \gamma_n(i))\, d\lambda$$

$$= \int_{\mathbb{N}} \liminf_n \phi_n(i) d\lambda + \int_{\mathbb{N}} \lim_n \gamma_n(i)\, d\lambda$$

$$= \int_{\mathbb{N}} \liminf_n \phi_n(i) d\lambda + \mathrm{M} \sum_{i=1}^{\infty} \delta_H(P_i)$$

$$= \int_{\mathbb{N}} \liminf_n \phi_n(i) d\lambda \mathrm{MT} \qquad (2.11)$$

for all i. Hence from (2.10) and (2.11) we also have

$$\int_{\mathbb{N}} \liminf_n \phi_n(i) d\lambda \leq \liminf_n \int_{\mathbb{N}} \phi_n(i) d\lambda$$

$$= \liminf_n \sum_{i=1}^{\infty} \left(H^{[P_i]} s^{(i)} \right)_n$$

$$= \liminf_n (Hs)_n. \qquad (2.12)$$

Now using Lemma 1

$$\int_{\mathbb{N}} \liminf_n \phi_n(i) d\lambda = \int_{\mathbb{N}} \liminf_n (H^{[P_i]} s^{(i)})\, d\lambda$$

$$\geq \int_{\mathbb{N}} \delta_H(P_i) \xi_i\, d\lambda$$

$$= \sum_{i=1}^{\infty} \delta_H(P_i)\, \xi_i. \qquad (2.13)$$

Hence by (2.12) and (2.13), we get

$$\sum_{i=1}^{\infty} \delta_H(P_i)\,\alpha_i \leq \liminf_n (Hs)_n$$

which completes the proof of (2.6).

If we replace s by -s in (2.6), we immediately obtain (2.7).

If $s^{(i)}$ is convergent for any $i \in \mathbb{N}$ then $\theta_i \coloneqq \xi_i = \sigma_i$ for any $i \in \mathbb{N}$. Hence our Theorem 4 generalizes Theorem 3.4 in [12]. Moreover, this theorem also tells us that the core of the sequence Hs does not exceed the interval $\left[\sum_{i=1}^{\infty} \delta_H(P_i)\,\xi_i, \sum_{i=1}^{\infty} \delta_H(P_i)\,\sigma_i\right]$.

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